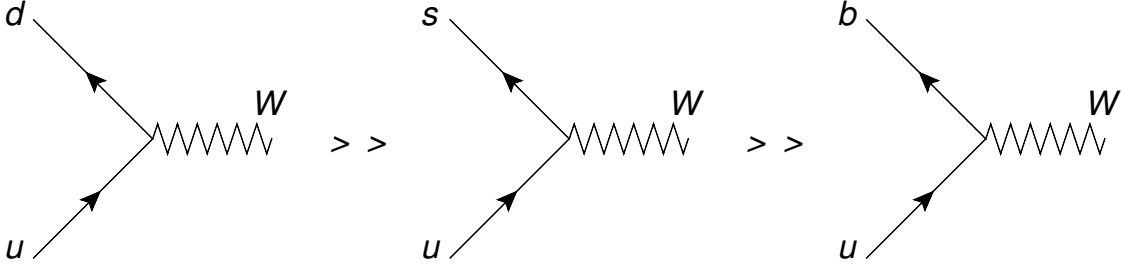


11 Family mixing and the Kobayashi-Maskawa matrix

The above discussion has been considerably simplified since it completely ignored mixing between the three fermion families. For example, if the coupling $u \rightarrow d + W^+$ is allowed the couplings $u \rightarrow s + W^+$ and $u \rightarrow b + W^+$ are not possible in the model. However it turns out that the transitions



are observed with the hierarchy as indicated. They can be summarised by defining a transition $u \rightarrow d'$ with d' a linear superposition of the d, s, b quarks. The quark states (u_i) and (d_i) with $i = 1, 2, 3$ for the three families constructed in the previous sections are eigenstates of the mass matrix but the charged electroweak transition is not diagonal in these states but rather it is diagonal in a (u'_i) , (d'_i) basis called the flavour basis. The two bases are related as follows:

$$u_{iL} = S_{ij}^{u_L} u'_{jL}, \quad d_{iL} = S_{ij}^{d_L} d'_{jL}, \quad u_{iR} = S_{ij}^{u_R} u'_{jR}, \quad d_{iR} = S_{ij}^{d_R} d'_{jR}, \quad i, j = 1, 2, 3. \quad (11.1)$$

Eq. (11.1) can be written in a matrix form $\mathbf{u}_L = \mathbf{S}^{u_L} \mathbf{u}'_L$ and similarly for the right-handed up sector as well as the left-handed and right-handed $down$ sectors. As will be shown later the matrices $\mathbf{S}^{u_L}, \mathbf{S}^{u_R}, \dots$ are unitary. The quark sector of the $SU(2)_L \otimes U(1)_Y$ lagrangian is written in general¹⁸ (see eq. (5.22))

$$\mathcal{L}_F = \sum_i (\overline{u'_{iL}} \overline{d'_{iL}}) i \not{D}_{qL} \begin{pmatrix} u'_{iL} \\ d'_{iL} \end{pmatrix} + \sum_i \overline{u'_{iR}} i \not{D}_{uR} u'_{iR} + \sum_i \overline{d'_{iR}} i \not{D}_{dR} d'_{iR}, \quad (11.2)$$

so that the electroweak interactions are diagonal in the “primed” flavour basis. After symmetry breaking, the most general Yukawa lagrangian takes the form in the “primed” basis, (see eqs. (8.26), (8.27)), ignoring for the moment the H boson couplings to the fermions

$$\begin{aligned} \mathcal{L}_Y &= - \frac{v}{\sqrt{2}} \sum_{ij} (\overline{u'_{iL}} c_{ij}^u u'_{jR} + \overline{d'_{iL}} c_{ij}^d d'_{jR} + \text{h.c.}) \\ &= - \frac{v}{\sqrt{2}} (\overline{\mathbf{u}'_L} \mathbf{C}_u \mathbf{u}'_R + \overline{\mathbf{d}'_L} \mathbf{C}_d \mathbf{d}'_R + \text{h.c.}) \end{aligned} \quad (11.3)$$

¹⁸We could in fact identify flavour and mass eigenstates of the up sector and take for simplicity $\mathbf{S}^{u_L} = \mathbf{S}^{u_R} = \mathbf{1}$.

where the complex, 3×3 , $\mathbf{C}_u, \mathbf{C}_d$ matrices are the generalized Yukawa couplings. Including the common normalisation factor $v/\sqrt{2}$ with the $\mathbf{C}_u$ and $\mathbf{C}_d$ matrices, the most general such matrices can be written as a product :

$$\frac{v}{\sqrt{2}}\mathbf{C}_u = \mathbf{M}_u \mathbf{T}_u, \quad \frac{v}{\sqrt{2}}\mathbf{C}_d = \mathbf{M}_d \mathbf{T}_d \quad (11.4)$$

with $\mathbf{M}_u$ a hermitian matrix ($\mathbf{M}_u = \mathbf{M}_u^\dagger$) and $\mathbf{T}_u$ a unitary matrix ($\mathbf{T}_u^{-1} = \mathbf{T}_u^\dagger$). The hermitian matrix can be diagonalized by a unitary transformation, $\mathbf{M}_u = \mathbf{S}_u^{-1} \mathbf{m}_u \mathbf{S}_u = \mathbf{S}_u^\dagger \mathbf{m}_u \mathbf{S}_u$ where $\mathbf{m}_u$ is diagonal with real eigenvalues, and similarly for the d sector. The Yukawa lagrangian reduces to the very simple diagonal form :

$$\begin{aligned} \mathcal{L}_Y &= -\bar{\mathbf{u}}_L \mathbf{m}_u \mathbf{u}_R - \bar{\mathbf{u}}_R \mathbf{m}_u \mathbf{u}_L - \bar{\mathbf{d}}_L \mathbf{m}_d \mathbf{d}_R - \bar{\mathbf{d}}_R \mathbf{m}_d \mathbf{d}_L \\ &= -\bar{\mathbf{u}} \mathbf{m}_u \mathbf{u} - \bar{\mathbf{d}} \mathbf{m}_d \mathbf{d} \end{aligned} \quad (11.5)$$

when written in terms of the mass eigenstate basis related to the original one by :

$$\begin{aligned} \mathbf{u}_L &= \mathbf{S}_u \mathbf{u}'_L, & \mathbf{u}_R &= \mathbf{S}_u \mathbf{T}_u \mathbf{u}'_R \\ \mathbf{d}_L &= \mathbf{S}_d \mathbf{d}'_L, & \mathbf{d}_R &= \mathbf{S}_d \mathbf{T}_d \mathbf{d}'_R, \end{aligned} \quad (11.6)$$

which defines the matrices $\mathbf{S}^{u_L}, \mathbf{S}^{u_R}, \dots$ introduced above. We remark that, as advertized before, the transformation from the primed basis to the unprimed one is unitary since such are the $\mathbf{S}, \mathbf{T}$ matrices. The components of $\mathbf{u}$ in which the mass matrix is diagonal are, by definition, the “physical” quark fields (u, c, t) of definite mass eigenstate (*idem* for the d sector). The same is obviously true for the Higgs couplings which are diagonal in the $\mathbf{u}$ and $\mathbf{d}$ bases. Having achieved a simple form for the Yukawa lagrangian we re-write now the gauge part $\mathcal{L}_F$ in terms of these physical fields. Singling out the neutral current interactions we have

$$\begin{aligned} \mathcal{L}_F(\text{neutral current}) &= \sum_i (\bar{u}'_{i_L} i \not{D}_{u_L} u'_{i_L} + \bar{u}'_{i_R} i \not{D}_{u_R} u'_{i_R}) + d' \text{ sector} \\ &= (\bar{\mathbf{u}}'_L i \not{D}_{u_L} \mathbf{u}'_L + \bar{\mathbf{u}}'_R i \not{D}_{u_R} \mathbf{u}'_R) + \mathbf{d}' \text{ sector}, \end{aligned} \quad (11.7)$$

in which we keep only the diagonal (in $SU(2)$) part of the operator $\not{D}_L$. Because of the unitarity of the transformations within the left-handed bases and the right-handed bases the above lagrangian immediately reduces itself to

$$\begin{aligned} \mathcal{L}_F(\text{neutral current}) &= \bar{\mathbf{u}}_L i \not{D}_{u_L} \mathbf{u}_L + \bar{\mathbf{u}}_R i \not{D}_{u_R} \mathbf{u}_R \\ &+ \bar{\mathbf{d}}_L i \not{D}_{u_L} \mathbf{d}_L + \bar{\mathbf{u}}_R i \not{D}_{u_R} \mathbf{u}_R. \end{aligned} \quad (11.8)$$

This equation is the basis for the slogan that, in the Standard Model and in agreement with experiments, there is “no flavour-changing neutral current”, in other words the neutral current is diagonal in flavour space as well as in the mass eigenstates : it does not induce transition between the c quark and the u quark or between the b quark and the d quark for example. The case of the charged current pieces is more involved because it couples the *up* sector and the *down* sector which do not transform with the same unitary matrices and, as a consequence, there is no reason for the charged current interactions to be diagonal in the basis which diagonalises the mass matrix. Indeed we have, from eqs. (11.2), (5.43):

$$\begin{aligned}\mathcal{L}_F(\text{quark charged current}) &= \frac{e}{\sqrt{2}\sin\theta_W} (\bar{\mathbf{u}}'_L \mathcal{W}^* \mathbf{d}'_L + \bar{\mathbf{d}}'_L \mathcal{W} \mathbf{u}'_L) \\ &= \frac{e}{\sqrt{2}\sin\theta_W} (\bar{\mathbf{u}}_L \mathcal{W}^* \mathbf{S}_u \mathbf{S}_d^\dagger \mathbf{d}_L + \bar{\mathbf{d}}_L \mathbf{S}_d \mathbf{S}_u^\dagger \mathcal{W} \mathbf{u}_L) \quad (11.9)\end{aligned}$$

where the last equation is obtained from eq. (11.6). The matrix $\mathbf{CKM} = \mathbf{S}_u \mathbf{S}_d^\dagger$ is the famous Cabibbo-Kobayashi-Maskawa matrix which parameterises the flavour changing content of charged current transitions, *i.e* transitions between left-handed **up** spinors (u_L, c_L, t_L) of definite mass to **down** spinors (d_L, s_L, b_L) of definite mass. Its matrix elements are often written as

$$\mathbf{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \quad (11.10)$$

The matrix $\mathbf{CKM}$ is unitary, since both $\mathbf{S}_u, \mathbf{S}_d^\dagger$ are such, and its matrix elements must satisfy relations of type $V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0$ or $V_{ub}^* V_{ub} + V_{cb}^* V_{cb} + V_{tb}^* V_{tb} = 1$. The determination of the V_{ij} is a very active area of particle physics phenomenology at present and it is one of the aims of the LHCb collaboration at CERN and Belle II at KEK in Japan. If V_{ud} is mainly constrained from nuclear β decays the others are essentially determined from K decays and heavy flavour decays. The 2018 edition of the particle data group¹⁹ quotes the following values

$$\begin{aligned}|V_{ud}| &= 0.97446 \pm 0.00010, & |V_{us}| &= 0.22452 \pm 0.00044, & |V_{ub}| &= (3.65 \pm 0.12)10^{-03} \\ |V_{cd}| &= 0.22438 \pm 0.00044, & |V_{cs}| &= 0.97359^{+0.00010}_{-0.00011}, & |V_{cb}| &= (4.214 \pm 0.076)10^{-02} \\ |V_{td}| &= (8.96^{+0.24}_{-0.23})10^{-03}, & |V_{ts}| &= (4.133 \pm 0.074)10^{-02}, & |V_{tb}| &= 0.999105 \pm 0.000032\end{aligned} \quad (11.11)$$

The $\mathbf{CKM}$ matrix generalizes to three families the Cabibbo angle, $\sin\theta_C = \lambda \sim .22 \simeq V_{us}$ introduced long ago to deal with the mixing of two families. The Cabibbo-Kobayashi-Maskawa quark mixing

¹⁹ M. Tanabashi *et. al.* (Particle Data Group), Phys. Rev. **D98** (2018) 030001 (<http://pdg.lbl.gov>).

matrix is a 3×3 unitary matrix and from its definition²⁰ it depends on nine parameters which can be chosen as three real parameters and six phases. Changing the phase of u_{iL} and d_{jL} respectively by $e^{i\phi_{ui}}$ and $e^{i\phi_{dj}}$ the **CKM** matrix elements $(\mathbf{S}_u \mathbf{S}_d^\dagger)_{ij}$ becomes $e^{i(\phi_{dj}-\phi_{ui})}(\mathbf{S}_u \mathbf{S}_d^\dagger)_{ij}$. The five arbitrary phase differences can be used to absorb as many phases of the **CKM** matrix leaving one $\mathcal{CP}$ violating phase. One should of course shift the phases of the right-handed fields and the left-handed ones by the same amount to leave the mass terms eq. (11.5) invariant. The independent parameters of the **CKM** matrix are chosen as three angles ($c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$ with $0 < \theta_{ij} < \pi/2$) and a phase δ ($0 < \delta < 2\pi$) and one writes:

$$\begin{aligned} \mathbf{CKM} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \cdot \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \cdot \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \end{aligned} \quad (11.12)$$

This last form is not very illuminating and in view of the relative smallness of $|s_{13}| \simeq |V_{ub}| \simeq 3.57 \cdot 10^{-03}$, $|s_{23}| \simeq |V_{cb}| = 4.11 \cdot 10^{-02}$ the approximation $c_{13} \approx c_{23} \approx 1$ is justified. With $|s_{12}| \approx |V_{us}| = 0.225$, Wolfenstein introduced the convenient and often used parameterisation (see the PDG review¹⁹):

$$\mathbf{CKM} = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}, \quad (11.13)$$

with

$$\begin{aligned} s_{12} &\approx \lambda = 0.22453 \pm 0.00044, & A &= 0.836 \pm 0.0015 \\ \rho &= 0.122^{+0.018}_{-0.017}, & \eta &= 0.355^{+0.012}_{-0.011} \end{aligned} \quad (11.14)$$

This parameterisation shows that the charged current transition, for example, of a u quark to d, s, b quarks takes place with amplitudes which are proportional to $(1 - \lambda^2/2)$, λ , $A\lambda^3(\rho - i\eta)$ respectively. The phase factor η , or equivalently δ in eq. (11.12), is responsible for $\mathcal{CP}$ violation in the Standard Model (see appendix B, in particular B.3). The measurement of this $\mathcal{CP}$ violating parameter, in kaon and B meson systems, for example, is of great theoretical interest in order to understand the origin of $\mathcal{CP}$ violation and of great practical importance since it may be related to the origin of the baryon asymmetry in the universe.

²⁰A unitary matrix $\mathbf{U}$ can be written $\mathbf{U} = \exp(i \sum_a \alpha^a T^a)$, with T^a the generators of the $SU(3)$ group for $a = 1, \dots, 8$ and $T^9 = \mathbb{I}$. The α^a are real parameters. A matrix $\mathbf{U}$ parameterised as in eq. (11.12) is often written $\mathbf{U} = U_{23}U_{13}(\delta)U_{12}$.

A very important point is to check experimentally the unitarity of the **CKM** matrix : indeed any violation of one of the unitarity relations may indicate the existence of a new quark or a new family of quarks. Present data are consistent with the unitarity of the **CKM** matrix within a 3% accuracy.