12 Neutrinos and the Pontecorvo-Maki-Nakagawa-Sakata matrix

The absence of a **CKM** mixing matrix for the leptonic sector requires a comment. We assumed that the right-handed neutrinos decouple from the observed world. As a consequence, as mentioned above, the neutrinos ν_e, ν_μ, ν_τ remain massless even after spontaneous symmetry breaking since there are, in the lagrangian density, no terms coupling left-handed and right-handed fields like in eq. (8.26). Therefore no mass matrix can be constructed from which the "physical" neutrino states are defined. When studying the weak-current transition from charged leptons to neutrinos we are thus free to define the neutrino physical states as those for which the charged weak current is diagonal in lepton flavour.

However, recent experiments have shown that neutrinos oscillate *i.e.* they change flavour when propagating from their emission point to their detection point. This can be explained if neutrinos are massive. If one follows the same procedure as for the quarks one introduces right-handed fields and this leads to Dirac type massive neutrinos. There is another possibility which relies on the fact that, being neutral, neutrinos can be their own antiparticles and this leads to Majorana type neutrinos. In the first case the total lepton number $L = L_e + L_\mu + L_\tau$ is conserved, while, in the latter case, it is not. In this section, we deal with Dirac neutrinos, the Majorana case being treated in sec. 15.

We assume that, like quarks, the neutrinos are of Dirac type with both left-handed and right-handed components. In the flavour basis, besides the triplet of left-handed fields $\boldsymbol{\nu}'_L$ one introduces a triplet of right-handed fields $\boldsymbol{\nu}'_R$, singlets under $SU(2)_L$,

$$\boldsymbol{\nu}'_L = \begin{pmatrix} \nu_{e_L} \\ \nu_{\mu_L} \\ \nu_{\tau_L} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\nu}'_R = \begin{pmatrix} \nu_{e_R} \\ \nu_{\mu_R} \\ \nu_{\tau_R} \end{pmatrix}. \tag{12.1}$$

In this notation ν_{e_L} is the neutrino produced by an electron in a charged current interaction and similarly for ν_{μ_L} and ν_{τ_L}. Assuming the $SU(2)_L \otimes U(1)_Y$ symmetry holds true, the right handed neutrinos cannot be produced or interact in reactions mediated by gauge bosons: they do not couple to the $SU(2)_L$ gauge bosons nor to the $U(1)_Y$ boson since being neutral $y_{\nu_R} = 2e_{\nu_R} = 0$, thus they do not couple to $W^\pm, Z$ or γ gauge bosons. They can be produced in Higgs decays but, given that the Higgs couplings are proportional to the masses, it is fair to assert that the production rate via this channel is not measurable. Their only effect is to give masses to neutrinos. The charged current transition (eq. (5.28)) is given by the term $(g/\sqrt{2})\,(\overline{\boldsymbol{e}}_L \not{W}\, \boldsymbol{\nu}'_L + \overline{\boldsymbol{\nu}'_L} \not{W}^* \boldsymbol{e}_L)$, diagonal in flavour where

$$\boldsymbol{e}_L = \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix} \tag{12.2}$$

is the triplet of left-handed charged leptons. We emphasize that e_L, μ_L, τ_L are the mass eigenstates of the charged leptons. In analogy with the case of quarks, after symmetry breaking, the Dirac mass term for neutrinos is of the form

$$\mathcal{L}_{Y_D} = -\,\frac{v}{\sqrt{2}}\,\overline{\boldsymbol{\nu}'_L}\,\boldsymbol{C}_\nu\,\boldsymbol{\nu}'_R + \text{ h.c.} \tag{12.3}$$

Following the steps leading from eq. (11.3) to eq. (11.6), we introduce the notation ($\mathbf{M}_\nu$ hermitian, $\mathbf{T}_\nu$ unitary),

$$\frac{v}{\sqrt{2}}\mathbf{C}_\nu = \mathbf{M}_\nu\ \mathbf{T}_\nu, \tag{12.4}$$

and diagonalize the hermitian matrix by the transformation $\mathbf{M}_\nu = \mathbf{S}_\nu^{-1}\mathbf{m}_\nu\mathbf{S}_\nu = \mathbf{S}_\nu^\dagger\mathbf{m}_\nu\mathbf{S}_\nu$ ($\mathbf{S}_\nu$ unitary). Defining

$$\boldsymbol{\nu}_L = \boldsymbol{S}_\nu\,\boldsymbol{\nu}'_L \qquad \text{and} \qquad \boldsymbol{\nu}_R = \boldsymbol{S}_\nu\,\boldsymbol{T}_\nu\,\boldsymbol{\nu}'_R, \tag{12.5}$$

the Yukawa lagrangian becomes diagonal,

$$\Rightarrow \mathcal{L}_{Y_D} = -\,\overline{\boldsymbol{\nu}}_L\,\boldsymbol{m}_\nu\,\boldsymbol{\nu}_R - \overline{\boldsymbol{\nu}}_R\,\boldsymbol{m}_\nu\,\boldsymbol{\nu}_L = -\,\overline{\boldsymbol{\nu}}\ \boldsymbol{m}_\nu\,\boldsymbol{\nu} \tag{12.6}$$

with m_1, m_2, m_3 the three real eigenvalues of $\boldsymbol{m}_\nu$ and ν_1, ν_2, ν_3 the three neutrino mass eigenstates

$$\boldsymbol{\nu} = \boldsymbol{\nu}_L + \boldsymbol{\nu}_R = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \tag{12.7}$$

Then, using eq. (12.5), the charged current transition is written

$$\begin{aligned} \mathcal{L}(\text{leptonic charged current}) &= \frac{g}{\sqrt{2}}\,(\overline{\boldsymbol{e}}_L \not{W}\,\boldsymbol{\nu}'_L + \overline{\boldsymbol{\nu}'_L} \not{W}^*\,\boldsymbol{e}_L) \\ &= \frac{e}{\sqrt{2}\,\sin\theta_W}\,(\overline{\boldsymbol{e}}_L \not{W}\,\boldsymbol{S}_\nu^\dagger\,\boldsymbol{\nu}_L + \overline{\boldsymbol{\nu}}_L\,\boldsymbol{S}_\nu \not{W}^*\,\boldsymbol{e}_L). \end{aligned} \tag{12.8}$$

Similarly to the **CKM** mixing matrix one introduces the Pontecorvo-Maki-Nakagawa-Sakata matrix[21] $\mathbf{PMNS} = \boldsymbol{S}_\nu^\dagger$ the matrix elements of which are often written as[22]:

$$\mathbf{PMNS} \;=\; \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \tag{12.9}$$

[21] In 1952, B. Pontevorvo was the first to mention the possibility of $\nu_e - \overline{\nu}_e$ oscillations. In 1962, the year when ν_μ was discovered, Ziro Maki, Masami Nakagawa and Shoichi Sakata, assuming two kinds of neutrinos proposed a "particle mixture theory of neutrino", Prog. Theor. Physics **28** (1962), 870.

[22] The **PMNS** matrix appears simpler than the **CKM** one since the gauge interaction eq. (12.8) is written directly in terms of the charged lepton mass eigenstates.

where e, μ, τ refer to flavour states and 1, 2, 3 to mass eigenstates.

It is easy to check that all terms in the lagrangian density are invariant under the global phase change of all fields

$$\boldsymbol{e}_L \to e^{i\lambda}\boldsymbol{e}_L, \quad \boldsymbol{\nu}_L \to e^{i\lambda}\boldsymbol{\nu}_L, \quad \boldsymbol{\nu}_R \to e^{i\lambda}\boldsymbol{\nu}_R. \tag{12.10}$$

To this invariance corresponds the conservation of the total lepton number defined as $L = \sum_\alpha L_\alpha$, $\alpha = e, \mu, \tau$. It follows that such transitions as $\mu^- \to e^- + \gamma$ or $\mu^- \to e^- + e^+ + e^-$ are allowed in the model. A recent fit to available data shows that the mixing pattern is quite different from that of the quark[23]

$$\begin{aligned}
&|U_{e1}| = 0.800 \to 0.844, \quad |U_{e2}| = 0.515 \to 0.581, \quad |U_{e3}| = 0.139 \to 0.155\\
&|U_{\mu1}| = 0.229 \to 0.516, \quad |U_{\mu2}| = 0.438 \to 0.699, \quad |U_{\mu3}| = 0.614 \to 0.790\\
&|U_{\tau1}| = 0.249 \to 0.528, \quad |U_{\tau2}| = 0.462 \to 0.715, \quad |U_{\tau3}| = 0.595 \to 0.776.
\end{aligned} \tag{12.11}$$

As for quarks, no satisfactory model can account for this mixing pattern.

The phenomenology of neutrino mixing is discussed below in the framework of Dirac neutrinos. There are several recent reviews on this topic, in particular by Nakamura and Petcov[24] and by Giganti, Lavignac and Zito[25]. The case of Majorana neutrinos is discussed in sec. 15 and by Bilenky and Petcov[26].

12.1 Neutrino survival and oscillation

The space-time evolution of a state of given mass is ($\hbar = c = 1$)

$$|\nu_i(x)> = \mathrm{e}^{-i(E\,t - \vec{k}\,\vec{x})}|\nu_i> . \tag{12.12}$$

As will be seen below it is justified to assume the neutrinos to be ultrarelativistic particles so that $E \approx k + m_i^2/2\,k$ and the equation becomes (x denotes now the length travelled by the neutrino)

$$|\nu_i(x)> = \mathrm{e}^{-ix\,(m_i^2/2\,k)}|\nu_i> . \tag{12.13}$$

[23] I. Esteban, M.C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler, T. Schwetz, JHEP **1701** (2017) 087, arXiv:1611.01514 [hep-ph]. The variation of the coefficients is given for a 3 σ range.

[24] K. Nakamura, S.T. Petcov, in Particle Data Group (PDG), M. Tanabashi *et. al.*, Phys. Rev. **D98** (2018) 030001 (`http://pdg.lbl.gov`).

[25] C. Giganti, S. Lavignac, M. Zito, Prog. Part. Nucl. Phys. **98** (2018) 1, arXiv:1710.00715 [hep-ex].

[26] S.M. Bilenky, S.T. Petcov, Rev. Mod. Phys. **59** (1987) 671.

We consider a neutrino of type α produced in a charged current interaction with momentum k. It is a coherent superposition of neutrino states of definite mass[27],

$$|\nu_\alpha >= \sum_i U^*_{\alpha i} |\nu_i >, \qquad \alpha = e, \mu, \tau : \qquad i = 1, 2, 3. \tag{12.14}$$

The space-time evolution of this neutrino is given at a later time t and at a distance $x = t$ by

$$|\nu_\alpha(x) >= \sum_i U^*_{\alpha i} \, \mathrm{e}^{-ix\,(m_i^2/2\,k)} |\nu_i > . \tag{12.15}$$

The probability for this neutrino, initially of flavour α, to be observed as a neutrino of flavour β at the distance x from the emission point is

$$P(\nu_\alpha \to \nu_\beta) = | < \nu_\beta | \nu_\alpha(x) > |^2 \;\; = \;\; \sum_{i,j} U^*_{\alpha i} U_{\alpha j} U_{\beta i} U^*_{\beta j} \exp\left(i\, x \,\frac{\delta m^2_{ji}}{2k} \right), \tag{12.16}$$

where the symbol $\delta m^2_{ji} = m_j^2 - m_i^2$. Separating the real and imaginary part of the phase factor and using the unitarity of the U matrix ($U^*_{\alpha i} U_{\beta i} = \delta_{\alpha\beta}$) this expression can be written as[28]:

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4 \sum_{i>j} \mathrm{Re}(U^*_{\alpha i} U_{\alpha j} U_{\beta i} U^*_{\beta j}) \, \sin^2 \left(x \,\frac{\delta m^2_{ij}}{4k} \right) + 2 \sum_{i>j} \mathrm{Im}(U^*_{\alpha i} U_{\alpha j} U_{\beta i} U^*_{\beta j}) \sin \left(x \frac{\delta m^2_{ij}}{2k} \right), \tag{12.17}$$

For the time reversed transition $P(\nu_\beta \to \nu_\alpha)$ permuting α and β in eq. (12.16) is equivalent to permuting i and j so that it comes out

$$P(\nu_\beta \to \nu_\alpha) = \delta_{\alpha\beta} - 4 \sum_{i>j} \mathrm{Re}(U^*_{\alpha i} U_{\alpha j} U_{\beta i} U^*_{\beta j}) \, \sin^2 \left(x \,\frac{\delta m^2_{ij}}{4k} \right) - 2 \sum_{i>j} \mathrm{Im}(U^*_{\alpha i} U_{\alpha j} U_{\beta i} U^*_{\beta j}) \sin \left(x \frac{\delta m^2_{ij}}{2k} \right), \tag{12.18}$$

which exhibits the violation of $\mathcal{T}$ invariance due to the phase factor in the **PMNS** matrix. One finds the same result, eq. (12.18), for $P(\overline{\nu}_\alpha \to \overline{\nu}_\beta)$, exhibiting this time the $\mathcal{CP}$ violation of the model. Then $\mathcal{CPT}$ is conserved because $P(\nu_\beta \to \nu_\alpha) = P(\overline{\nu}_\alpha \to \overline{\nu}_\beta)$. Finally one has the sum rule, valid in the three family model

$$1 = \sum_{\nu_\beta = \nu_e, \nu_\mu, \nu_\tau} P(\nu_\alpha \to \nu_\beta) \quad = \sum_{\overline{\nu}_\beta = \overline{\nu}_e, \overline{\nu}_\mu, \overline{\nu}_\tau} P(\overline{\nu}_\alpha \to \overline{\nu}_\beta), \qquad \text{for any } \nu_\alpha, \overline{\nu}_\alpha. \tag{12.19}$$

It is important to remark that in case of a disappearance probability, $P(\nu_\alpha \to \nu_\alpha)$, the last term in the eqs. (12.17) or (12.18) disappears since terms such as $U^*_{\alpha i} U_{\alpha j} U_{\alpha i} U^*_{\alpha j}$ are real and therefore a disappearance probability cannot depend on the imaginary part of the **PMNS** matrix elements.

[27] From eqs. (12.5), (12.9) a flavour field ν_{α_L} is related to the fields ν_{i_L} of given masses by $\nu_{\alpha_L} = \sum_i (S^\dagger)_{\alpha i} \nu_{i_L} = \sum_i U_{\alpha i} \nu_{i_L}$, but the state $|\nu_\alpha >$ is created by the field $\overline{\nu}_{\alpha_L}$, hence eq. (12.14).

[28] We use $\cos\left(x \delta m^2_{ij}/2k\right) = 1 - 2\,\sin^2\left(x\,\delta m^2_{ij}/4k\right)$, the factor 1 then leading to the $\delta_{\alpha\beta}$ term.

12.2 Summary of results

It turns out, as will be discussed below, that the last factor in eqs. (12.17) and (12.18) is small. Then, oscillations, as a function of x, in the probability for the neutrino to change flavour (or to remain in the same flavour) are essentially induced by the factors $\sin^2(x\,\delta m_{ij}^2/4k)$. For the oscillation to be seen this factor should be of $\mathcal{O}(1)$. To be quantitative, we have to inject the $\hbar$ and c factors to make the argument of the $\sin^2$ factor dimensionless. One finds[29]

$$\frac{\delta m_{ij}^2\, x}{4\,k} \;\Rightarrow\; 1.27\,10^{-18}\frac{\delta m_{ij}^2\,[\mathrm{GeV}^2]\; x\,[\mathrm{km}]}{k\,[\mathrm{GeV}]} = 1.27\;\frac{\delta m_{ij}^2\,[\mathrm{eV}^2]\; x\,[\mathrm{km}]}{k\,[\mathrm{GeV}]} = 1.27\;\frac{\delta m_{ij}^2\,[\mathrm{eV}^2]\; x\,[\mathrm{m}]}{k\,[\mathrm{MeV}]}, \quad (12.20)$$

where we have given this expression in terms of the units commonly used. One defines the oscillation length associated to a given mass squared difference by the condition

$$\frac{\delta m_{ij}^2\, x}{4\,k} = \pi \qquad \Rightarrow \qquad x\,[\mathrm{m}] = 2.47\,\frac{k\,[\mathrm{MeV}]}{\delta m_{ij}^2\,[\mathrm{eV}^2]} \quad \text{or} \quad x\,[\mathrm{km}] = 2.47\,\frac{k\,[\mathrm{GeV}]}{\delta m_{ij}^2\,[\mathrm{eV}^2]}. \qquad (12.21)$$

Conversely, we can use this formula to estimate the sensitivity of typical neutrino experiments to mass squared differences as shown in the table below. In some experimental conditions, a factor $(x\,\delta m_{ij}^2/4k)$

Source	type of ν	k [GeV]	x [km]	$<\boldsymbol{\delta}\mathbf{m^2}>$ [$\mathbf{eV^2}$]
Reactors (short baseline)	$\overline{\nu}_e$	10^{-3}	1	$\mathbf{10^{-3}}$
Reactors (long baseline)	$\overline{\nu}_e$	10^{-3}	100	$\mathbf{10^{-5}}$
Accelerators (short baseline)	$\nu_\mu, \overline{\nu}_\mu$	1	1	$\mathbf{1}$
Accelerators (long baseline)	$\nu_\mu, \overline{\nu}_\mu$	1	10^3	$\mathbf{10^{-3}}$
Atmospheric	$\nu_e, \overline{\nu}_e, \nu_\mu, \overline{\nu}_\mu$	1	10^4	$\mathbf{10^{-4}}$
Sun	ν_e	10^{-2}	$1.5\,10^8$	$\mathbf{10^{-10}}$

Table 1: *Sensitivity in terms of δm_{ij}^2 of the different types of neutrino experiments characterised by the energy k of the neutrino and the distance x between the ν source and the detector.*

may remain small and its contribution to the oscillation pattern becomes negligible. In other cases on the contrary, it stays very large and the oscillating $\sin^2$ term averages out to 1/2. These facts simplify the analysis of the oscillations as will be seen below in the discussion of several experiments. We give here the values of the parameters, with the **PMNS** matrix written as in eq. (11.12), obtained from of

[29] In eq. (12.12) the dimensionless phase should be $-i(Et - \vec{k}\,\vec{x})/\hbar$, with the energy E measured in GeV and k in GeV/c as appropriate for neutrino experiments. It can be written $-i(Ect - \vec{k}\,\vec{x})/(\hbar c)$, with both E and k as well as the mass measured in GeV and $[ct] = [x]$ in km if c is expressed in km/sec. We have (see the PDG tables) $\hbar c = 197.3267\,10^{-21}$ GeV·km; using the approximate form eq. (12.13), the oscillation factor in eq. (12.17) becomes $\delta m_{ij}^2\,[\mathrm{GeV}^2]\; x\,[\mathrm{km}]/(4\,k\,[\mathrm{GeV}]\;\hbar c[\mathrm{GeV\cdot km}]) = 1.27\,10^{-18}\delta m_{ij}^2\,[\mathrm{GeV}^2]\; x\,[\mathrm{km}]/k\,[\mathrm{GeV}]$.

a recent global analysis of data[30]

$$\boxed{\begin{array}{c} \delta m^2_{21} = (6.92 - 7.91)\, 10^{-5}\, \mathrm{eV}^2, \qquad \delta m^2_{31} = (2.392 - 2.594)\, 10^{-3}\, \mathrm{eV}^2 \\ \sin^2\theta_{12} = 0.265 - 0.346, \quad \sin^2\theta_{23} = 0.430 - 0.602, \quad \sin^2\theta_{13} = 0.0190 - 0.0239. \end{array}} \tag{12.22}$$

By convention the mass m_2 is chosen larger than m_1 but there are two possibilities for m_3: either $m_1 < m_2 < m_3$, labeled *normal hierarchy*, or $m_3 < m_1 < m_2$, labeled *inverted hierarchy*. The above results are obtained assuming a *normal hierarchy*. In the other case the values of the parameters are very similar except of course for the sign of δm^2_{3i}. For example, the best fit value for δm^2_{32} is $2.418\ 10^{-3}\, \mathrm{eV}^2$ (normal hierarchy) and $\delta m^2_{32} = -2.478\ 10^{-3}\, \mathrm{eV}^2$ (inverted hierarchy). There are two independent δm^2_{ij} and given the relative smallness of δm^2_{21}, one is justified to take $|\delta m^2_{32}| \approx |\delta m^2_{31}|$. Concerning the mass hierarchy and the $\mathcal{CP}$ violating phase δ, they are difficult to extract because, as will be seen, they enter the observables with small coefficients. One notes that the angle θ_{13} is much smaller than the other mixing angles and small θ_{13} approximations will often be used. At present fits to data seem to indicate a value $\delta \approx 3\pi/2$, with large error bars, for both mass hierarchies and a preference for normal hierarchy.

One does not know the absolute scale of neutrino masses. If we assume $m_1 \ll m_2$ one gets $m_2 \approx 8.6\, 10^{-3}\, \mathrm{eV}$ and $m_3 \approx 5.1\, 10^{-2}\, \mathrm{eV}$, while in the inverted hierarchy case, assuming $m_3 \ll m_1$ the result is $m_1 \approx m_2 \approx 5.1\ 10^{-2}\, \mathrm{eV}$. One way to experimentally access the mass scale of neutrinos is through nuclear β decay which allows to give a direct limit on the $\overline{\nu}_e$ mass. For these purposes, several past and ongoing experiments study tritium decay[31], ${}^3H \to {}^3He + e^- + \overline{\nu}_e$. The electron energy spectrum is sensitive to the neutrino mass near the upper end of the spectrum. Denoting E_0 the total energy release in the decay, the maximum value of the electron energy is $E < E_0 \approx 18$ keV, if the neutrino is massless. A non vanishing neutrino mass will slightly reduce the bound to $E < E_0 - m_{\nu_e}$ and will modify the shape of the spectrum near this end point of the distribution. Near the end point the electron spectrum behaves as

$$dN/dE \propto E_{\nu_e} k_{\nu_e} = (E_0 - E)((E_0 - E)^2 - m^2_{\nu_e})^{1/2}, \tag{12.23}$$

which has a non-zero derivative, in fact $-\infty$, if the neutrino is massive. The effect is very hard to measure since the rate of energetic electrons is very low. A limit established some years ago[32] was

[30] F. Capozzi, E. Lisi, A. Marrone, A. Palazzo, Prog. Part. Nucl. Phys. **102** (2018) 48, arXiv:1804.09678 [hep-ph]; see also P.F. de Salas, S. Gariazzo, O. Mena, C.A. Ternes, M. Tórtola, Front. Astron. Space Sci. **5** (2018) 36; arXiv:1806.11051 [hep-ph]; P.F. de Salas, D.V. Forero, C.A. Ternes, M. Tórtola, J.W.F. Valle, Phys. Lett. **B782** (2018) 633, arXiv:1708.01186 [hep-ph]; NuFIT webpage, `http://www.nu-fit.org/`.

[31] G. Drexlin, V. Hannen, S. Mertens, and C. Weinheimer, Adv.High Energy Phys. (2013) 293986, arXiv:1307.0101.

[32] Troitzk Collaboration, V.N. Aseev *et al.* Phys. Rev **D84** (2011) 112003.

$m_{\nu_e} < 2.05$ eV at 95% c.l., quite a bit higher than the tentative scales suggested above. The KATRIN experiment which started operation in 2018, in Karlsruhe, quotes now an upper limit of 1.1 eV at 90% c.l.[33]. By 2024 the collaboration expects to reach 0.2 eV (90% c.l.) or 0.35 eV (5 σ). Finally astrophysical and cosmological limits are available on the sum of neutrino masses and a recent result reported by the Planck collaboration is[34]

$$\sum_j m_j < .12 \text{ eV}, \tag{12.24}$$

but this result is model dependent. In the following we use for the **PMNS** matrix, eq. (12.9), the representation, eq. (11.12).

12.3 Survival probabilities in vacuum

Since, in experiments, both survival and oscillation probabilities can be measured we quote below the general form of these expressions for 3 flavoured neutrinos[35] In the analysis of results it will turn out that different approximations can be made, depending on the experimental set-up, which simplify considerably the general expressions. The reduced forms will be easily obtained from the results given in this section and the next. The simplest case is the electron survival probability, the exact expression of which is:

$$\begin{aligned} P(\nu_e \to \nu_e) = 1 \; - \; & \sin^2(2\theta_{12})\cos^4(\theta_{13})\sin^2\left(x\frac{\delta m^2_{21}}{4k}\right) - \sin^2(2\theta_{13})\sin^2(\theta_{12})\sin^2\left(x\frac{\delta m^2_{32}}{4k}\right) \\ - \; & \sin^2(2\theta_{13})\cos^2(\theta_{12})\sin^2\left(x\frac{\delta m^2_{31}}{4k}\right), \end{aligned} \tag{12.25}$$

insensitive to δ (thus $P(\nu_e \to \nu_e) = P(\overline{\nu}_e \to \overline{\nu}_e)$) and to the hierarchy of mass. The muon survival is given by:

$$\begin{aligned} P(\nu_\mu \to \nu_\mu) = 1 \; - \; & [\sin^2(2\theta_{12})\cos^4(\theta_{23}) + \sin^2(2\theta_{23})[\cos^4(\theta_{12}) + \sin^4(\theta_{12})]\sin^2(\theta_{13})] \;\sin^2\left(x\frac{\delta m^2_{21}}{4k}\right) \\ - \; & [\sin^2(2\theta_{23})\cos^2(\theta_{13})\cos^2(\theta_{12}) + \sin^2(2\theta_{13})\sin^4(\theta_{23})\sin^2(\theta_{12})] \;\sin^2\left(x\frac{\delta m^2_{32}}{4k}\right) \\ - \; & [\sin^2(2\theta_{23})\cos^2(\theta_{13})\sin^2(\theta_{12}) + \sin^2(2\theta_{13})\sin^4(\theta_{23})\cos^2(\theta_{12})] \;\sin^2\left(x\frac{\delta m^2_{31}}{4k}\right) \\ - \; & 8\,J\,\cos(\delta)\,\mathrm{COS}_{\nu_\mu} + \mathcal{O}(\sin^3(\theta_{13})), \end{aligned} \tag{12.26}$$

[33] G. Drexin for the KATRIN Collaboration, 16th TAUP International Conference, Toyama, Japan, Sept. 2019.

[34] Planck 2018 results. VI. Cosmological parameters, N. Aghanim *et al., in* Astronomy and & Astrophysics, 2018, arXiv:1807.06209 [astro-ph.CO].

[35] Exact expressions in a somewhat different form are found in V. Barger, D. Marfatia, K. Whisnant, Int. J. Mod. Phys. **E12** (2003) 569, hep-ph:0308123.

with the Jarlskog factor[36] J:

$$J = \frac{1}{8}\sin(2\theta_{12})\sin(2\theta_{23})\sin(2\theta_{13})\cos(\theta_{13}), \quad (12.27)$$

and the expression COS_{ν_μ}:

$$\mathrm{COS}_{\nu_\mu} = \left[\cos^2(\theta_{23})\cos(2\theta_{12})\sin^2\left(x\frac{\delta m^2_{21}}{4k}\right) - \sin^2(\theta_{23})\left(\sin^2\left(x\frac{\delta m^2_{32}}{4k}\right) - \sin^2\left(x\frac{\delta m^2_{31}}{4k}\right)\right)\right] \quad (12.28)$$

More precisely, in eq. (12.26), we have neglected very small terms of type $\sin^4(\theta_{13})\sin^2(x\delta m^2_{21}/4k)$, $\sin^3(\theta_{13})\cos(\delta)$ and $\sin^2(\theta_{13})\cos^2(\delta)$. The τ survival probability is obtained from this equation, by exchanging $\sin^2(\theta_{23})$ and $\cos^2(\theta_{23})$ and reversing the sign of the $\cos(\delta)$ term.

12.4 Oscillation in vacuum, $\mathcal{CP}$ asymmetries, mass hierarchy and δ

It is important to obtain the dependence on the phase δ of the oscillation probabilities as it is related to the $\mathcal{CP}$ asymmetries and to the mass hierarchy. In fact, all oscillation probabilities have, up to a sign, the same dependence on $\sin(\delta)$ which is relatively easy to obtain. Injecting the parameterisation eq. (11.12) in the U matrices in eq. (12.16) one finds without approximations

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= \sin^2(2\theta_{12})\cos^2(\theta_{13})[\cos^2(\theta_{23}) - \sin^2(\theta_{23})\sin^2(\theta_{13})]\sin^2\left(x\frac{\delta m^2_{21}}{4k}\right) \\ &+ \sin^2(2\theta_{13})\sin^2(\theta_{23})\sin^2(\theta_{12})\sin^2\left(x\frac{\delta m^2_{32}}{4k}\right) \\ &+ \sin^2(2\theta_{13})\sin^2(\theta_{23})\cos^2(\theta_{12})\sin^2\left(x\frac{\delta m^2_{31}}{4k}\right) \\ &+ 4\,J\cos(\delta)\ \mathrm{COS} + 2\,J\sin(\delta)\ \mathrm{SIN}, \end{aligned} \quad (12.29)$$

with

$$\mathrm{COS} = \left[\cos(2\theta_{12})\sin^2\left(x\frac{\delta m^2_{21}}{4k}\right) - \sin^2\left(x\frac{\delta m^2_{32}}{4k}\right) + \sin^2\left(x\frac{\delta m^2_{31}}{4k}\right)\right] \quad (12.30)$$

$$\mathrm{SIN} = \left[\sin\left(x\frac{\delta m^2_{21}}{2k}\right) + \sin\left(x\frac{\delta m^2_{32}}{2k}\right) + \sin\left(x\frac{\delta m^2_{13}}{2k}\right)\right]. \quad (12.31)$$

[36] C. Jarlskog, Z. Phys. **C29** (1985) 491. Due to the unitarity of the **PMNS** matrix one shows that $\mathrm{Im}(U^*_{\alpha i}U_{\alpha j}U_{\beta i}U^*_{\beta j})$ with $\alpha \neq \beta$, $\alpha,\beta = e$, μ, τ, $i \neq j$, $i,j = 1,2,3$, is up to a sign an invariant, thus $\mathrm{Im}(U^*_{e3}U_{e2}U_{\mu 3}U^*_{\mu 2}) = J\sin(\delta)$.

Using the sum rule eq. (12.19) or by direct calcultation it comes out:

$$\begin{aligned}
P(\nu_e \to \nu_\tau) &= \sin^2(2\theta_{12})\cos^2(\theta_{13})[\sin^2(\theta_{23}) - \cos^2(\theta_{23})\sin^2(\theta_{13})]\sin^2\left(x\frac{\delta m_{21}^2}{4k}\right) \\
&+ \sin^2(2\theta_{13})\cos^2(\theta_{23})\sin^2(\theta_{12})\sin^2\left(x\frac{\delta m_{32}^2}{4k}\right) \\
&+ \sin^2(2\theta_{13})\cos^2(\theta_{23})\cos^2(\theta_{12})\sin^2\left(x\frac{\delta m_{31}^2}{4k}\right) \\
&- 4\,J\cos(\delta)\ \mathrm{COS} - 2\,J\sin(\delta)\ \mathrm{SIN},
\end{aligned} \tag{12.32}$$

From the expressions given in eqs. (12.25) to (12.32) and with the help of the relations given in sec. 12.1 we can obtain all survival or oscillation probabilities of neutrinos and antineutrinos. For instance, one obtains $P(\nu_\mu \to \nu_e)$ from $P(\nu_e \to \nu_\mu)$ by reversing the sign of δ in eq. (12.29) and one derives $P(\nu_\mu \to \nu_\tau) = 1 - P(\nu_\mu \to \nu_e) - P(\nu_\mu \to \nu_\mu)$ from the sum rule. For completeness we quote it at the same level of approximation as the previous rates with the further simplification of dropping all terms proportional to $\sin^2(\theta_{13})$ in the first line:

$$\begin{aligned}
P(\nu_\mu \to \nu_\tau) &= -\frac{1}{4}\sin^2(2\theta_{23})\sin^2(2\theta_{12})\sin^2\left(x\frac{\delta m_{21}^2}{4k}\right) \\
&+ \sin^2(2\theta_{23})[\cos^2(\theta_{12}) - \sin^2(\theta_{12})\sin^2(\theta_{13})]\cos^2(\theta_{13})\sin^2\left(x\frac{\delta m_{32}^2}{4k}\right) \\
&+ \sin^2(2\theta_{23})[\sin^2(\theta_{12}) - \cos^2(\theta_{12})\sin^2(\theta_{13})]\cos^2(\theta_{13})\sin^2\left(x\frac{\delta m_{31}^2}{4k}\right) \\
&+ 4\,J\cos(\delta)\ \mathrm{COS}_\tau + 2\,J\sin(\delta)\ \mathrm{SIN},
\end{aligned} \tag{12.33}$$

with

$$\mathrm{COS}_\tau = \cos(2\theta_{23})\left[\cos(2\theta_{12})\sin^2\left(x\frac{\delta m_{21}^2}{4k}\right) + \sin^2\left(x\frac{\delta m_{32}^2}{4k}\right) - \sin^2\left(x\frac{\delta m_{31}^2}{4k}\right)\right] \tag{12.34}$$

If one defines a measure of the $\mathcal{CP}$ asymmetry in the oscillation $\nu_\alpha \to \nu_\beta$ by

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = P(\nu_\alpha \to \nu_\beta) - P(\overline{\nu}_\alpha \to \overline{\nu}_\beta), \tag{12.35}$$

then the following relations hold true:

$$\boxed{\begin{aligned}
\mathcal{A}(\nu_e \to \nu_\mu) = \ -\mathcal{A}(\nu_\mu \to \nu_e) \ = \ -\mathcal{A}(\nu_e \to \nu_\tau) \ = \ \mathcal{A}(\nu_\mu \to \nu_\tau) \ &= 4\,J\sin(\delta)\,\mathrm{SIN} \\
= 4\,J\sin(\delta)\left[\sin\left(x\frac{\delta m_{21}^2}{2k}\right) + \sin\left(x\frac{\delta m_{32}^2}{2k}\right) + \sin\left(x\frac{\delta m_{13}^2}{2k}\right)\right]&.
\end{aligned}} \tag{12.36}$$

Since the δm_{ij}^2 factors are not independent, $\delta m_{31}^2 = \delta m_{32}^2 + \delta m_{21}^2$, one can eliminate m_{31}^2, for example, and obtain:

$$\begin{aligned} \mathrm{SIN} &= 4\,\sin\left(x\frac{\delta m_{21}^2}{4k}\right)\sin\left(x\frac{\delta m_{31}^2}{4k}\right)\sin\left(x\frac{\delta m_{32}^2}{4k}\right) && (12.37)\\ &= 4\,\sin\left(x\frac{\delta m_{21}^2}{4k}\right)\sin^2\left(x\frac{\delta m_{32}^2}{4k}\right)+\mathcal{O}\left(\sin^2\left(x\frac{\delta m_{21}^2}{4k}\right)\right). && (12.38)\end{aligned}$$

where the last relation is valid when $x\,\delta m_{21}^2/4k$ is small compared to $x\,\delta m_{32}^2/4k$. Coming back to the oscillation probabilities, the coefficient of the $\cos(\delta)$ piece in the equations can likewise be simplified and one finds[37]

$$\begin{aligned} \mathrm{COS} &= 2\,\sin\left(x\frac{\delta m_{21}^2}{4k}\right)\sin\left(x\frac{\delta m_{31}^2}{4k}\right)\cos\left(x\frac{\delta m_{32}^2}{4k}\right)-2\,\sin^2(\theta_{12})\sin^2\left(x\frac{\delta m_{21}^2}{4k}\right) && (12.39)\\ &= 2\,\sin\left(x\frac{\delta m_{21}^2}{4k}\right)\sin\left(x\frac{\delta m_{32}^2}{4k}\right)\cos\left(x\frac{\delta m_{32}^2}{4k}\right)+\mathcal{O}\left(\sin^2\left(x\frac{\delta m_{21}^2}{4k}\right)\right). && (12.40)\end{aligned}$$

Under these simplifications[38], and neglecting small $\sin^2(\theta_{13})$ corrections in the coefficients of terms in $\sin^2\left(x\,\delta m_{21}^2/4k\right)$, the oscillation probabilities for $\nu_e \to \nu_\mu$ and $\nu_e \to \nu_\tau$ take the form:

$$\begin{aligned} P(\nu_e\to\nu_\mu) \approx{}& \sin^2(2\theta_{12})\cos^2(\theta_{23})\sin^2\left(x\frac{\delta m_{21}^2}{4k}\right)+\sin^2(2\theta_{13})\sin^2(\theta_{23})\sin^2\left(x\frac{\delta m_{32}^2}{4k}\right)\\ &+8\,J\,\sin\left(x\frac{\delta m_{21}^2}{4k}\right)\sin\left(x\frac{\delta m_{32}^2}{4k}\right)\left[\cos(\delta)\cos\left(x\frac{\delta m_{32}^2}{4k}\right)+\sin(\delta)\sin\left(x\frac{\delta m_{32}^2}{4k}\right)\right] && (12.41)\\ P(\nu_e\to\nu_\tau) \approx{}& \sin^2(2\theta_{12})\sin^2(\theta_{23})\sin^2\left(x\frac{\delta m_{21}^2}{4k}\right)+\sin^2(2\theta_{13})\cos^2(\theta_{23})\sin^2\left(x\frac{\delta m_{32}^2}{4k}\right)\\ &-8\,J\,\sin\left(x\frac{\delta m_{21}^2}{4k}\right)\sin\left(x\frac{\delta m_{32}^2}{4k}\right)\left[\cos(\delta)\cos\left(x\frac{\delta m_{32}^2}{4k}\right)+\sin(\delta)\sin\left(x\frac{\delta m_{32}^2}{4k}\right)\right] && (12.42)\end{aligned}$$

and similarly for other probabilities. The difference between normal and inverted hierarchy occurs only in the sign of the $\cos(\delta)$ coefficient, all other terms being insensitive to the sign of δm_{32}^2. If the present experimental value of δ around $3\pi/2$ (with large error bars) is confirmed, it will be very difficult to solve the mass hierarchy problem from oscillation experiments in vacuum. More on this later.

Sometimes, it is sufficient to consider only a two neutrino system, ν_e and ν_x say, in which case the

[37] One has also $\mathrm{COS}_\tau = -2\cos(2\theta_{23})\left[\sin\left(x\delta m_{21}^2/4k\right)\sin\left(x\delta m_{32}^2/4k\right)\cos\left(x\delta m_{31}^2/4k\right)+\sin^2(\theta_{12})\sin^2\left(x\delta m_{21}^2/4k\right)\right]$.

[38] They are particularly useful in oscillation experiments with accelerator neutrinos. Note that one can use indifferently δm_{32}^2 or δm_{31}^2 in eqs. (12.38) and (12.40).

oscillation formulae simplify considerably:

$$\begin{aligned} P(\nu_e \to \nu_e) &= 1 - \sin^2(2\theta_{12}) \sin^2\left(x \frac{\delta m_{21}^2}{4k} \right) \\ P(\nu_e \to \nu_x) &= \sin^2(2\theta_{12}) \sin^2\left(x \frac{\delta m_{21}^2}{4k} \right) \end{aligned} \tag{12.43}$$