13 Neutrinos interactions with matter

Atmospheric (anti)neutrinos observed after crossing the earth or neutrinos produced in the sun propagate through matter and interact with it before reaching the detector. The scattering on protons, neutrons and electrons in matter will modify the oscillation patterns. This is the Mikheyev-Smirnov-Wolfenstein (MSW) effect[39]. The important parameters in this effect are the electron density in matter and the neutrino energy. For some values of the parameters large resonance effects enhance the neutrino conversion rate compared to what is expected in vacuum.

To illustrate this point it is sufficient to consider a two-flavour model with mass eigenvectors $|\nu_1>$ and $|\nu_2>$ with the 1 state being the lightest one. From eq. (12.13) the evolution of the doublet of $|\nu_i(t)>$ states, in vacuum, is given by ($t = x$):

$$i\frac{d}{dt}\begin{pmatrix}|\nu_1(t)>\\|\nu_2(t)>\end{pmatrix} = \mathcal{H}_0\begin{pmatrix}|\nu_1(t)>\\|\nu_2(t)>\end{pmatrix} = \begin{pmatrix}m_1^2/2k & 0\\0 & m_2^2/2k\end{pmatrix}\begin{pmatrix}|\nu_1(t)>\\|\nu_2(t)>\end{pmatrix}, \tag{13.1}$$

with $\mathcal{H}_0$ the free hamiltonian. A global phase change on the $|\nu_i(t)>$ states does not affect the physics but shifts the hamiltonian by a matrix proportional to the unit matrix. For instance, a phase change $im_1^2t/2k$ on both states leads to the evolution equation:

$$i\frac{d}{dt}\begin{pmatrix}|\nu_1(t)>\\|\nu_2(t)>\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & \delta m^2/2k\end{pmatrix}\begin{pmatrix}|\nu_1(t)>\\|\nu_2(t)>\end{pmatrix}, \tag{13.2}$$

with $\delta m^2 = m_2^2 - m_1^2$ taken to be positive. The evolution of the flavour states $|\nu_e(t)>$ and $|\nu_x(t)>$ ($|\nu_x(t)>$ can be a combination of $|\nu_\mu(t)>$ and $|\nu_\tau(t)>$)[40], is easily obtained from the relation:

$$\begin{pmatrix}|\nu_e(t)>\\|\nu_x(t)>\end{pmatrix} = \mathcal{R}(\theta)\begin{pmatrix}|\nu_1(t)>\\|\nu_2(t)>\end{pmatrix} \qquad \text{with the matrix } \mathcal{R}(\theta) = \begin{pmatrix}\cos(\theta) & \sin(\theta)\\-\sin(\theta) & \cos(\theta)\end{pmatrix}. \tag{13.3}$$

We then have:

$$i\frac{d}{dt}\begin{pmatrix}|\nu_e(t)>\\|\nu_x(t)>\end{pmatrix} = \mathcal{R}(\theta)\begin{pmatrix}0 & 0\\0 & \delta m^2/4k\end{pmatrix}\mathcal{R}^T(\theta)\begin{pmatrix}|\nu_e(t)>\\|\nu_x(t)>\end{pmatrix} \tag{13.4}$$

$$= \begin{pmatrix}(\delta m^2/2k)\sin^2(\theta) & (\delta m^2/4k)\sin(2\theta)\\(\delta m^2/4k)\sin(2\theta) & (\delta m^2/2k)\cos^2(\theta)\end{pmatrix}\begin{pmatrix}|\nu_e(t)>\\|\nu_x(t)>\end{pmatrix} = \mathcal{H}_0^{\text{fl}}\begin{pmatrix}|\nu_e(t)>\\|\nu_x(t)>\end{pmatrix}, \tag{13.5}$$

with $\mathcal{H}_0^{\text{fl}}$ is the free hamiltonian in the flavour basis. The interaction of neutrinos with matter can preserve or destroy the coherence of the system. In the latter case, the state of the particles (momentum and spin) is modified and it can be shown that incoherent interactions are negligible.

[39] L. Wolfenstein, Phys. Rev. **D17** (1978) 2369; S.P. Mikheyev, A.Yu. Smirnov, Prog. Part. Nucl. Phys. **23** (1989) 41.

[40] We have in mind solar neutrinos but the discussion applies to any two flavour system.

13.1 Incoherent scattering

For example, for neutrinos up to a GeV, scattering on nucleons $\nu_x + n \to x + p$ is the dominant process and the cross section[41] can be parameterised as:

$$\sigma \approx 10^{-43} \left(\frac{E_\nu}{\mathrm{MeV}}\right)^2 \mathrm{cm}^2, \tag{13.6}$$

with E_ν the energy in the frame where the nucleon is at rest. The scattering length of the neutrino in matter is $l_{\mathrm{matter}} = 1/N_N\sigma$ where N_N is the number of nucleons per cm^3. In the core of the sun, the density is 150 gr/cm^3, so approximately 10^{26} nucleons per cm^3. The scattering length is then:

$$l_{\mathrm{sun}} \;\approx\; 10^{17} \left(\frac{E_\nu}{\mathrm{MeV}}\right)^{-2} \mathrm{cm} \;\approx\; 10^{12} \left(\frac{E_\nu}{\mathrm{MeV}}\right)^{-2} \mathrm{km}. \tag{13.7}$$

The typical energy of solar neutrinos being .1 MeV $< E_\nu <$ 10 MeV, the corresponding scattering length is 10^{14} km $> l_{\mathrm{sun}} > 10^{10}$ km, to be compared to the sun radius of $7\,10^5$ km. Incoherent neutrino scattering in the sun is negligible.

The range of energy of neutrinos crossing the earth is much larger, from .1 MeV for solar neutrinos to TeV's for atmospheric or cosmic ones. At high energy the charged current ν-nucleon cross section behaves as

$$\sigma \approx 6.7\ 10^{-39} \left(\frac{E_\nu}{\mathrm{GeV}}\right) \mathrm{cm}^2.$$

The matter density in the earth ranges from 4 gr/cm^3 in the mantle to, on the average, 11 gr/cm^3 in the core. This leads respectively to $N_N = 2.4\ 10^{24}$ to $6.6\ 10^{24}$ nucleons per cm^3. Then, the scattering length of 100 GeV neutrinos l_{earth} varies from 6. 10^6 km in the mantle to 2 10^6 km in the inner core. This is to be compared to the mantle thickness of 2.9 10^3 km and the core radius of 3.4 10^3 km. Thus the effect of the earth matter is negligible for neutrinos of energy up to hundreds of GeV. On the contrary, for neutrinos around 100 TeV and above the earth becomes opaque since the cross section grows linearly with energy.

13.2 Coherent scattering

Coherence of the neutrino system is preserved by forward elastic scattering of the neutrino on matter. This can go via neutral current interactions, on protons, neutrons or electrons, $\nu_{e,x} + N \to \nu_{e,x} + N$ and $\nu_{e,x} + e^- \to \nu_{e,x} + e^-$, which are universal for all neutrinos species or via charged current exchange which is specific to ν_e scattering on electrons (see fig. 15 in sec. 14.4). These interactions add a piece

[41] J.A. Formaggio, G.P. Zeller, Rev. Mod. Phys. **84** (2012) 1307.

to the hamiltonian which becomes

$$\mathcal{H} = \mathcal{H}_0^{\rm fl} + \mathcal{H}_{\rm int}^{\rm fl} \tag{13.8}$$

where $\mathcal{H}_{\rm int}^{\rm fl}$ is diagonal in flavour. Implementing a phase change on the states amounts to shifting the hamiltonian by a matrix proportional to unity and one can thus subtract the universal neutral current contribution leaving the charged current one which affects only the element $<\nu_e|\mathcal{H}_{\rm int}^{\rm fl}|\nu_e>= <\nu_e|\mathcal{H}_{\rm cc}^{\rm fl}|\nu_e>$. This interaction is given by eqs. (2.1), (2.2) in sec. 2.1,

$$\begin{aligned}\frac{G_F}{\sqrt{2}}\,\overline{\psi}_e(x)\gamma_\mu(1-\gamma_5)\psi_{\nu_e}(x)\,\overline{\psi}_{\nu_e}(x)\gamma^\mu(1-\gamma_5)\psi_e(x) &= 2\sqrt{2}\,G_F\,\overline{\psi}_{e_L}(x)\gamma_\mu\psi_{\nu_{e_L}}(x)\,\overline{\psi}_{\nu_{e_L}}(x)\gamma^\mu\psi_{e_L}(x)\\ &= 2\sqrt{2}\,G_F\,\overline{\psi}_{\nu_{e_L}}(x)\gamma^\mu\psi_{\nu_{e_L}}(x)\,\overline{\psi}_{e_L}(x)\gamma_\mu\psi_{e_L}(x),\end{aligned} \tag{13.9}$$

where a Fierz transformation has been made to obtain the second line. The effective interaction hamiltonian of the neutrinos in matter is obtained by summing over all electrons in matter[42]:

$$\begin{aligned}\mathcal{H}_{\rm cc}^{\rm fl} &= 2\sqrt{2}\,G_F\int dp_e^3 f(p_e)\,<e_L(p_e)|\overline{\psi}_{\nu_{e_L}}(x)\gamma^\mu\psi_{\nu_{e_L}}(x)\,\overline{\psi}_{e_L}(x)\gamma_\mu\psi_{e_L}(x)|e_L(p_e)>\\ &= 2\sqrt{2}\,G_F\,\overline{\psi}_{\nu_{e_L}}(x)\gamma^\mu\psi_{\nu_{e_L}}(x)\int dp_e^3 f(p_e)\,<e_L(p_e)|\overline{\psi}_{e_L}(x)\gamma_\mu\psi_{e_L}(x)|e_L(p_e)>,\end{aligned} \tag{13.10}$$

where the electron energy distribution $f(p_e)$ in matter is homogeneous, isotropic and is normalised to $\int dp_e^3 f(p_e) = 1$. Assuming the electron approximately at rest in the medium, the space components γ_i can be neglected and the combinations $\overline{\psi}\gamma_\mu\psi$ reduce to $\overline{\psi}\gamma_0\psi = \psi^\dagger\psi$, so that

$$\begin{aligned}\mathcal{H}_{\rm cc}^{\rm fl} &= 2\sqrt{2}\,G_F\,\psi^\dagger_{\nu_{e_L}}(x)\psi_{\nu_{e_L}}(x)\int dp_e^3 f(p_e)\,<e_L(p_e)|\psi^\dagger_{e_L}(x)\psi_{e_L}(x)|e_L(p_e)>,\\ &= \sqrt{2}\,G_F\,\psi^\dagger_{\nu_{e_L}}\psi_{\nu_{e_L}}\,N_e,\end{aligned} \tag{13.11}$$

with N_e the density of electrons in the medium ($N_{e_L} = N_e/2$). The evolution equation will then be of the form

$$i\frac{d}{dt}\begin{pmatrix}|\nu_e(t)>\\|\nu_x(t)>\end{pmatrix} = \begin{pmatrix}(\delta m^2/2k)\sin^2(\theta)+\sqrt{2}\,G_F N_e & (\delta m^2/4k)\sin(2\theta)\\ (\delta m^2/4k)\sin(2\theta) & (\delta m^2/2k)\cos^2(\theta)\end{pmatrix}\begin{pmatrix}|\nu_e(t)>\\|\nu_x(t)>\end{pmatrix} = \mathcal{H}\begin{pmatrix}|\nu_e(t)>\\|\nu_x(t)>\end{pmatrix}. \tag{13.12}$$

Due to the charged current interaction the mass eigenstates $|\nu_i(t)>$ of eq. (13.1) no longer diagonalize the hamiltonian. Let us denote ω_1 and ω_2 the eigenvalues of the above matrix and $|\nu_{m_1}(t)>$ and $|\nu_{m_2}(t)>$ the corresponding mass eigenstates related to the flavour states $|\nu_e(t)>$ and $|\nu_x(t)>$ at time t by

$$\begin{aligned}|\nu_e(t)> &= \cos\theta^m|\nu_{m_1}(t)> + \sin\theta^m|\nu_{m_2}(t)>\\ |\nu_x(t)> &= -\sin\theta^m|\nu_{m_1}(t)> + \cos\theta^m|\nu_{m_2}(t)>\,.\end{aligned} \tag{13.13}$$

[42]M.C. Gonzalez-Garcia, M. Maltoni, Phys. Rep. **460** (2008) 1.

13.3 Matter of constant density

If N_e is independent of t, so are θ^m and the eigenvalues given by:

$$
\begin{aligned}
\omega_{1,2} &= \frac{G_F N_e}{\sqrt{2}} + \frac{\delta m^2}{4k} \mp \frac{1}{2}\sqrt{(\sqrt{2}G_F N_e - \cos(2\theta)\delta m^2/2k)^2 + (\sin(2\theta)\delta m^2/2k)^2} \\
&= \frac{\delta m^2}{4k}\left[\hat{A} + 1 \mp \sqrt{\sin^2(2\theta) + (\cos(2\theta) - \hat{A})^2}\right], \qquad (13.14)
\end{aligned}
$$

with ω_i the eigenvalue of the state $|\nu_{m_i}(t)>$. The important parameter $\hat{A}$ is defined by:

$$
\boxed{\hat{A} = \frac{2\sqrt{2}G_F k N_e}{\delta m^2},} \qquad (13.15)
$$

which is the ratio of the interaction energy in matter to the vacuum energy. The matrix $\mathcal{H}$ in eq. (13.12) is diagonalised by $\mathcal{R}^T(\theta^m)\,\mathcal{H}\,\mathcal{R}(\theta^m) = \mathrm{diag}(\omega_1, \omega_2)$ (see eq. (13.3)) and one finds:

$$
\tan(\theta^m) = \frac{\hat{A} - \cos(2\theta) + \sqrt{\sin^2(2\theta) + (\cos(2\theta) - \hat{A})^2}}{\sin(2\theta)} \qquad (13.16)
$$

from which we derive (for δm^2 positive):

$$
\boxed{
\begin{aligned}
\cos(2\theta^m) &= \frac{\cos(2\theta) - \hat{A}}{\sqrt{\sin^2(2\theta) + (\cos(2\theta) - \hat{A})^2}} \\
\sin(2\theta^m) &= \frac{\sin(2\theta)}{\sqrt{\sin^2(2\theta) + (\cos(2\theta) - \hat{A})^2}},
\end{aligned}
} \qquad (13.17)
$$

To obtain the oscillation probabilities we use eqs. (12.43):

$$
\boxed{P(\nu_e \to \nu_e) = 1 - \sin^2(2\theta^m)\,\sin^2\left(\frac{\delta M^2 t}{4k}\right), \qquad P(\nu_e \to \nu_x) = \sin^2(2\theta^m)\,\sin^2\left(\frac{\delta M^2 t}{4k}\right),} \qquad (13.18)
$$

where[43]

$$
\boxed{\delta M^2 = \delta m^2 \sqrt{\sin^2(2\theta) + (\cos(2\theta) - \hat{A})^2}.} \qquad (13.19)
$$

The corresponding oscillation length in matter is given by (see eq. (12.21):

$$
l_{\mathrm{mat}} = \frac{4\pi k}{\delta M^2} = \frac{2\pi}{\omega_2 - \omega_1} \qquad (13.20)
$$

Several cases can be distinguished assuming N_e constant in the medium (with δm^2 positive).

[43] The physics depends only on the difference $\omega_2 - \omega_1$ and θ^m, which are functions of the difference of the diagonal elements of $\mathcal{H}$, in agreement with the fact that one can modify $\mathcal{H}$ by adding to it a matrix proportional to unity.

- If $\hat{A} \ll 1$, then $\sin(2\theta^m) \approx \sin(2\theta)(1 + \hat{A}\cos(2\theta))$, $\delta M^2 \approx \delta m^2(1 - \hat{A}\cos(2\theta))$: the interaction with matter is small and the neutrino system evolves almost as in empty space, $l_{\mathrm{mat}} \approx l_{\mathrm{vac}}$ with a small correction;

- If $\hat{A} \gg |\cos(2\theta)|$, interaction with matter is dominant: then $\sin(2\theta^m) \approx \sin(2\theta)/\hat{A} \approx 0$ and $\cos(2\theta^m) \approx -1$, hence $\theta^m \approx \pi/2$: from eq. (13.13) the electron neutrino tends to a pure mass eigenstate $|\nu_{m_2}>$, the heaviest state ($\omega_2 \approx \sqrt{2}G_F N_e$); it propagates without oscillations independent of the value of the mixing angle in vacuum;

- If $\hat{A} \approx \cos(2\theta)$, this is the resonant regime: it occurs only if $\cos(2\theta)$ is positive ($0 < \theta < \pi/4$), then $\cos(2\theta^m) \approx 0$, $\sin(2\theta^m) \approx 1$, $\theta^m \approx \pi/4$, $l_{\mathrm{mat}} \approx l_{\mathrm{vac}}/\sin(2\theta)$; the electron neutrino is an equal combination of $|\nu_{m_1}>$ and $\nu_{m_2}>$, independent of the initial mixing angle, the amplitude of oscillations is maximal, since $\sin(2\theta^m) \approx 1$, as well as the oscillation length. For $\pi/4 < \theta < \pi/2$ there is no resonance effect possible and θ^m is always larger than $\pi/4$.

Remarks

- When applying eq. (13.11) to antineutrinos states one will obtain an extra $-$ sign[44], thus giving a contribution $-\sqrt{2}\,G_F N_e$ to $\mathcal{H}$. Then, the sign of $\hat{A}$ for antineutrinos is opposite to that for neutrinos. If the resonance condition $\hat{A} \approx \cos(2\theta)$ can be reached for neutrinos, it cannot occur for antineutrinos and vice-versa. For antineutrinos the resonance condition requires $\pi/4 < \theta < \pi/2$.

- The evolution of neutrinos in matter violates the $\mathcal{CP}$ symmetry, which is obvious since matter is not $\mathcal{CP}$ symmetric.

Application to solar neutrinos

Electron neutrinos are produced in the core of the sun where N_e can be as large as 6. 10^{25} cm^{-3}. It is useful to define the quantity N_{Res} by

$$N_{\mathrm{Res}} = \frac{\delta m^2 \cos(2\theta)}{2\sqrt{2}G_F k}, \tag{13.21}$$

related to the parameter $\hat{A}$ previously introduced by

$$\frac{N_e}{N_{\mathrm{Res}}} = \frac{\hat{A}}{\cos(2\theta)} \tag{13.22}$$

[44] $\bar{\nu}e \to \bar{\nu}e$ scattering is obtained from $\nu e \to \nu e$ by crossing symmetry which implies a relative - sign when crossing fermions.

Taking for θ and δm^2 the values θ_{12} and δm^2_{21} from eq. (12.22) below, one obtains

$$N^{21}_{\rm Res} \approx .8\ 10^{-6} \left(\frac{E_\nu}{\rm MeV}\right)^{-1} {\rm MeV}^3 \approx 10^{26} \left(\frac{E_\nu}{\rm MeV}\right)^{-1} {\rm cm}^{-3}, \tag{13.23}$$

so $N^{21}_{\rm Res} \lessapprox 10^{25}$ cm^{-3} for $E_\nu \gtrapprox 10$ MeV. In that case, the condition $N_e \gg N^{21}_{\rm Res}$ (equivalently $\hat{A} \gg \cos(2\theta_{12})$) is realised and the neutrino is produced in a mass eigenstate. On the contrary, neutrinos of energy $E_\nu \approx .1$ MeV evolve as in vacuum since they satisfy $N_e \ll N^{21}_{\rm Res}$. The range of values of θ_{12} given in eqs. (12.22), $30^{\rm o} < \theta_{12} < .38^{\rm o}$ implies $\cos(2\theta_{12}) > 0$ so that the resonance regime $\hat{A} \approx \cos(2\theta_{12})$ can be satisfied for neutrinos of intermediate energies. In the sun, however, N_e is a decreasing function of x, the distance from the center, and taking this effect into account requires a special treatment to which we turn in the next section. We can also consider oscillations to the third generation and estimate $N^{31}_{\rm Res}$. Using the values of θ_{13} and δm^2_{31} from eq. (12.22) one finds $N_e/N^{31}_{\rm Res} \approx 6.10^{-3}(E_\nu/{\rm MeV})$, so that $6.10^{-4} < N_e/N^{31}_{\rm Res} < 6.10^{-2}$ in the E_ν range $[.1, 10.]$ MeV, making matter effects negligible in this case. When studying oscillations in the sun, working in the 2 family oscillation model will be a good enough approximation.

Neutrinos through the earth

The electron density in the earth is much less than in the sun and it remains approximately constant in the core[45] ($N_e \approx 3.3\ 10^{24}$ cm^{-3}) and in the mantle ($N_e \approx 1.2\ 10^{24}$ cm^{-3}). It is then expected that solar neutrinos with $E_\nu < 10$ MeV will be little affected by coherent interactions when traversing the earth. However this will not the case for higher energy neutrinos in the GeV and multi-Gev range. Furthermore, in the 3-ν model, 13 oscillations will become important since $N_e/N^{31}_{\rm Res} = 2\sqrt{2}G_F k/(\delta m^2_{31}\cos(2\theta_{13}))$ can be of order 1 in the GeV range. This will be discussed later.

13.4 Matter of varying density: $\nu_{\rm e}$ in the sun

When the density of electrons decreases from the core to the surface, as it is the case in the sun, the angle $\theta^m(t)$ becomes a function of $x = t$. The variation of $\theta^m(x)$ should bring a $d\theta^m(x)/dx = \theta'_m(x)$ dependence in the evolution equations of the neutrino system. From eq. (13.13) written as

$$\begin{pmatrix} |\nu_e(x)\rangle \\ |\nu_x(x)\rangle \end{pmatrix} = \mathcal{R}(\theta^m(x)) \begin{pmatrix} |\nu_{m_1}(x)\rangle \\ |\nu_{m_2}(x)\rangle \end{pmatrix}, \tag{13.24}$$

[45] One assumes an equal number of neutrons and protons hence $N_p = N_e = N_N/2$, with N_N given above.

we derive

$$
\begin{aligned}
i\frac{d}{dx}\begin{pmatrix}|\nu_e(x)>\\|\nu_x(x)>\end{pmatrix} &= i\left(\frac{d}{dt}\mathcal{R}(\theta^m(x))\right)\begin{pmatrix}|\nu_{m_1}(x)>\\|\nu_{m_2}(x)>\end{pmatrix} + \mathcal{R}(\theta^m(x))\, i\frac{d}{dt}\begin{pmatrix}|\nu_{m_1}(x)>\\|\nu_{m_2}(x)>\end{pmatrix} \\
&= \mathcal{R}(\theta^m(x))\left[\mathcal{R}^T(\theta^m(x))\, i\left(\frac{d}{dt}(\mathcal{R}(\theta^m(x))\right) + \begin{pmatrix}\omega_1(x) & 0\\ 0 & \omega_2(x)\end{pmatrix}\right]\begin{pmatrix}|\nu_{m_1}(x)>\\|\nu_{m_2}(x)>\end{pmatrix} \\
&= \mathcal{R}(\theta^m(x))\begin{pmatrix}\omega_1(x) & i\theta'_m(x)\\ -i\theta'_m(x) & \omega_2(x)\end{pmatrix}\mathcal{R}^T(\theta^m(x))\begin{pmatrix}|\nu_e(x)>\\|\nu_x(x)>\end{pmatrix}, \qquad (13.25)
\end{aligned}
$$

similar to eq. (13.4) except for the off-diagonal term $i\theta'_m(x)$. If $|2\,\theta'_m(x)/(\omega_2(x)-\omega_1(x)| \ll 1$, then $\omega_1(x)$ and $\omega_2(x)$ will remain approximate eigenvalues of the system and the $|\nu_{m_i}(x)>$ will be approximately the mass eigenstates. Intuitively, one expects this to happen if the rate of change of the electron density $(1/N_e)dN_e/dx$ is very slow compared to the oscillation length in matter. This rate of change is measured by $(1/N_e)dN_e/dx = 1/r_0$, where a large value of r_0 corresponds to a small variation of N_e and if

$$r_0/l_{\rm mat} \gg 1, \qquad (13.26)$$

with $l_{\rm mat}$ given by eq. (13.20), then the variation of N_e will have a small effect on the neutrino mass eigenstates. More precisely, this condition is :

$$\frac{\omega_2(x)-\omega_1(x)}{2\,|\theta'_m(x)|} \gg 1. \qquad (13.27)$$

From eqs. (13.17) one derives

$$2\,\theta'_m(x) = \frac{d\hat{A}}{dx}\,\frac{\sin(2\theta)}{\sin^2(2\theta)+(\cos^2(2\theta)-\hat{A})^2}, \qquad (13.28)$$

and from eq. (13.15) one has,

$$\frac{d\hat{A}}{dx} = \frac{\hat{A}}{r_0}. \qquad (13.29)$$

Using then the relations

$$\frac{\sin^2(2\theta)+(\cos^2(2\theta)-\hat{A})^2}{\sin^2(2\theta)} = 1+\tan^{-2}(2\theta^m)\ , \qquad (13.30)$$

the condition (13.27) can be written:

$$\boxed{\frac{1}{\hat{A}}\,\frac{r_0\delta m^2}{2k}\sin^2(2\theta)(1+\tan^{-2}(2\theta^m))^{3/2} = \frac{2\pi r_0}{l_{\rm mat}}\,\frac{N_{\rm Res}}{N_e}\,\tan(2\theta)(1+\tan^{-2}(2\theta^m)) \gg 1\ .} \qquad (13.31)$$

If this condition is satisfied the evolution of the neutrino system in matter is said to be adiabatic. The flavoured neutrinos related, at the initial time, to the mass eigenstates $|\nu_{m_i}(x_0)>$ by the angle

$\theta^m = \theta^m(x_0)$ as in eq. (13.24), will be, at each point of the evolution, related to the mass eigenstates $|\nu_{m_i}(x)>$ by the angle $\theta^m(x)$, until they exit from matter in vacuum, at a distance R where the mixing angle is θ and the mass eigenstates $|\nu_i>$. The assumed adiabatic evolution does not mix the $|\nu_{m_1}(x)>$ and $|\nu_{m_2}(x)>$ states which evolve respectively to the $|\nu_1>$ and $|\nu_2>$ states of the vacuum when the neutrino exit from the medium. Thus, for

$$|\nu_e(x_0)> = \cos(\theta^m(x_0))|\nu_{m_1}(x_0)> + \sin(\theta^m(x_0))|\nu_{m_2}(x_0)>, \tag{13.32}$$

at some initial time, one has at time x,

$$|\nu_e(x)> = \cos(\theta^m(x))|\nu_{m_1}(x)> + \sin(\theta^m(x))|\nu_{m_2}(x)>, \tag{13.33}$$

and when the neutrino reaches the surface of the sun,

$$|\nu_e(R)> = \cos(\theta)|\nu_1> + \sin(\theta)|\nu_2>, \tag{13.34}$$

The probability to find a ν_e at the surface will be $|<\nu_e(R)|\nu_e(x_0)>|^2$, *i.e.*:

$$\begin{aligned} P(\nu_e \to \nu_e; x_0, R) &= [\cos(\theta)\cos(\theta^m(x_0)) <\nu_1|\nu_{m_1}(x_0)> + \sin(\theta)\sin(\theta^m(x_0)) <\nu_2|\nu_{m_2}(x_0)>]^2 \\ &= \frac{1}{2}[1 + \cos(2\theta)\cos(2\theta^m(x_0))] \ + \ \text{oscillating term} \\ &\approx \sin^2(\theta) + \cos(2\theta)\cos^2(\theta^m(x_0)), \end{aligned} \tag{13.35}$$

where we have supposed that the oscillating term averages out to 0. As a special case, if at x_0 the neutrino is produced in a pure mass eigenstate $|\nu_{m_2}(x_0)>$ ($\theta^m(x_0) = \pi/2$), then the neutrino will remain in this pure mass eigenstate $|\nu_{m_2}(x)>$ during its propagation until it reaches the surface where $|\nu_{m_2}(R)> = |\nu_2>$ in vacuum. The probability to find a ν_e at the surface will then be

$$\boxed{P(\nu_e \to \nu_e; x_0, R) = \sin^2(\theta).} \tag{13.36}$$

On the contrary, one may consider the extreme non-adiabaticity case of the evolution in matter: in that case a ν_e produced in the $|\nu_{m_2}(x_0)>$ state ends up as the $|\nu_1(R)>$ when exiting from the medium, and if this occurs

$$P(\nu_e \to \nu_e; x_0, R) = \cos^2(\theta). \tag{13.37}$$

in contrast with eq. (13.36). The general treatment of a non adiabatic evolution is given by Petcov[46]. It is easy to check that, in the sun, the adiabaticity condition is satisfied.

[46] S.T. Petcov, Phys. Lett. **200** (1988) 373.

13.5 Neutrinos through the earth

As mentioned above, for energetic neutrinos traversing the earth $N_e/N^{21}_{\rm Res}$ is very large and $N_e/N^{31}_{\rm Res}$ may be of order 1 for $E_\nu \gtrapprox 1$ GeV: indeed, in that case, $N_e \approx 1.2$ to $3.3\ 10^{24}$ cm^{-3} compared $N^{21}_{\rm Res} \approx 10^{23}\,(E_\nu/{\rm GeV})^{-1}$ cm^{-3} and $N^{31}_{\rm Res} \approx 10^{25}\,(E_\nu/{\rm GeV})^{-1}$ cm^{-3}. It is then necessary to work with the full 3-ν model. The free hamiltonian when acting on the mass eigenstates is

$$\mathcal{H}_0 = \begin{pmatrix} m_1^2/2k & 0 & 0 \\ 0 & m_2^2/2k & 0 \\ 0 & 0 & m_3^2/2k \end{pmatrix} . \tag{13.38}$$

After a change of phase on the states it can be put in the form

$$\mathcal{H}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2/2k & 0 \\ 0 & 0 & \delta m_{31}^2/2k \end{pmatrix} \tag{13.39}$$

with $\delta m_{ij}^2 = m_i^2 - m_j^2$. Going to the flavour basis,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \mathcal{U} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} , \tag{13.40}$$

the hamiltonian is written $\mathcal{H}_0^{\rm fl} = \mathcal{U}\,\mathcal{H}_0\mathcal{U}^\dagger$ where $\mathcal{U}$ is parameterised[20] as in eq. (11.12), $\mathcal{U} = U_{23}U_{13}(\delta)U_{12}$. Since the interaction in matter affects only the electron the interacting hamiltonian is written

$$\begin{aligned}
\mathcal{H}^{\rm fl} &= \mathcal{U} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2/2k & 0 \\ 0 & 0 & \delta m_{31}^2/2k \end{pmatrix} \mathcal{U}^\dagger + \begin{pmatrix} \sqrt{2}G_F N_e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= U_{23}\,U(\delta) \left[U_{13}U_{12} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2/2k & 0 \\ 0 & 0 & \delta m_{31}^2/2k \end{pmatrix} U_{12}^\dagger U_{13}^\dagger + \begin{pmatrix} \sqrt{2}G_F N_e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] U^\dagger(\delta)\,U_{23}^\dagger .
\end{aligned} \tag{13.41}$$

Several comments are in order. The matrix U_{23} does not affect the interaction matrix which can then be multiplied by U_{23} on the left and $U_{23}^\dagger$ on the right. Furthermore, writing $U_{13}(\delta) = U(\delta)\,U_{13}\,U^\dagger(\delta)$ with

$$U(\delta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} , \tag{13.42}$$

the δ dependence can be factored out as indicated above. We know that $\delta m_{21}^2 \ll \delta m_{31}^2$ and we have seen that, in the earth, for neutrinos in the GeV range and above, the ratio $\delta m_{21}^2/2\sqrt{2}G_F N_e k$ is very small which justifies the approximation $\delta m_{21}^2 = 0$ which is now done. This will considerably simplify the discussion[47]. The hamiltonian in the flavour basis can then be written:

$$\mathcal{H}^{\rm fl} = U_{23}\, U(\delta) \left[U_{13} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta m_{31}^2/2k \end{pmatrix} U_{13}^\dagger + \begin{pmatrix} \sqrt{2} G_F N_e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] U^\dagger(\delta)\, U_{23}^\dagger. \tag{13.43}$$

The matrix U_{12} plays no role because of our choice $\delta m_{21}^2 = 0$, so we take $\theta_{12} = 0$, $U_{12} = 1$. Then this equation becomes:

$$\mathcal{H}^{\rm fl} = U_{23}\, U(\delta) \begin{pmatrix} (\delta m_{31}^2/2k)\sin^2(\theta_{13}) + \sqrt{2} G_F N_e & 0 & (\delta m_{31}^2/4k)\sin^2(2\theta_{13}) \\ 0 & 0 & 0 \\ (\delta m_{31}^2/4k)\sin^2(2\theta_{13}) & 0 & (\delta m_{31}^2/2k)\cos^2(\theta_{13}) \end{pmatrix} U^\dagger(\delta)\, U_{23}^\dagger. \tag{13.44}$$

The diagonalisation of the interacting hamiltonian follows the procedure of sec. 13.3. Here one eigenvalue ω_2 is 0 while the other two, $\omega_{1,3}$, are given by

$$\omega_{1,3} \quad = \quad \frac{\delta m_{31}^2}{4k} \left[\hat{A} + 1 \mp \sqrt{\sin^2(2\theta_{13}) + (\cos(2\theta_{13}) - \hat{A})^2} \right], \tag{13.45}$$

identical to the eigenvalues given in eq. (13.14) with the substitution $\theta \to \theta_{13}$ and $\delta m^2 \to \delta m_{31}^2$. As in the work of M. Freund[47] $\hat{A}$ is now

$$\boxed{\hat{A} = 2\sqrt{2} G_F N_e k/\delta m_{31}^2}. \tag{13.46}$$

The 3×3 matrix in eq. (13.44) is diagonalised via the matrix U_{13}^m and $\mathcal{H}^{\rm fl}$ is then written:

$$\mathcal{H}^{\rm fl} = U_{23}\, U(\delta)\, U_{13}^m \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix} U_{13}^{m\dagger}\, U^\dagger(\delta)\, U_{23}^\dagger, \tag{13.47}$$

with the matrix U_{13}^m of the same form as U_{13} but function of the angle θ_{13}^m. This angle is given by eqs. (13.16) or (13.17) with the appropriate change of notation. Finally the matrix $\mathcal{U}^m$ which relates the flavour eigenstates and the mass eigenstates (with eigenvalues $\omega_1, 0, \omega_3$) of the interacting theory is of the usual form

$$\mathcal{U}^m = U_{23}^m\, U_{13}^m(\delta)\, U_{12}^m = U_{23}\, U_{13}^m(\delta)\, U_{12}, \tag{13.48}$$

[47] The full treatment, which is applied here in a simplified form, is given in M. Freund, Phys. Rev. **D64** (2001) 053003, [arXiv:hep-ph/0103300].

with

$$
\begin{aligned}
&\sin(\theta_{12}^m) = 0, \qquad \sin(\theta_{23}^m) = \sin(\theta_{23}), \qquad \delta^m = \delta \\
&\cos(2\theta_{13}^m) = \frac{\cos(2\theta_{13}) - \hat{A}}{\sqrt{\sin^2(2\theta_{13}) + (\cos(2\theta_{13}) - \hat{A})^2}} = \frac{\cos(2\theta_{13}) - \hat{A}}{\hat{C}} \\
&\sin(2\theta_{13}^m) = \frac{\sin(2\theta_{13})}{\sqrt{\sin^2(2\theta_{13}) + (\cos(2\theta_{13}) - \hat{A})^2}} = \frac{\sin(2\theta_{13})}{\hat{C}}.
\end{aligned}
\tag{13.49}
$$

with

$$
\hat{C} = \sqrt{\sin^2(2\theta_{13}) + (\cos(2\theta_{13}) - \hat{A})^2}\,. \tag{13.50}
$$

To reconstruct the various ν_e transition probabilities, one needs to define the oscillating factors given by $x(\omega_i - \omega_j)/2$. They are, in the small θ_{13} approximation (see eqs. (12.22)), and using $\hat{A} < 1$:

$$
\begin{aligned}
x\,\frac{(\omega_2 - \omega_1)}{2} &= -x\,\frac{\omega_1}{2} \approx -\hat{A}\frac{\delta m_{31}^2}{4k}\,x \\
x\,\frac{(\omega_3 - \omega_2)}{2} &= x\,\frac{\omega_3}{2} \approx \frac{\delta m_{31}^2}{4k}\,x \\
x\,\frac{(\omega_3 - \omega_1)}{2} &= x\,\hat{C}\,\frac{\delta m_{31}^2}{4k} \approx |1 - \hat{A}|\frac{\delta m_{31}^2}{4k}\,x.
\end{aligned}
\tag{13.51}
$$

The oscillation probabilities, eqs. (12.25), (12.29) and (12.32) considerably simplify because of the vanishing of θ_{12}: the only oscillating factor to be kept is $\sin^2(x\,(\omega_3 - \omega_1)/2) = \sin^2(x\,\hat{C}\,\delta m_{31}^2/4k)$ all others are multipled by $\sin(\theta_{12})$ and disappear. One finds:

$$
P(\nu_e \to \nu_\mu) \approx \sin^2(\theta_{23})\,\frac{\sin^2(2\theta_{13})}{\hat{C}^2}\,\sin^2\left(x\,\hat{C}\,\frac{\delta m_{31}^2}{4k}\right), \tag{13.52}
$$

Changing $\sin(\theta_{23})$ to $\cos(\theta_{23})$, one obtains $P(\nu_e \to \nu_\tau)$. In the small θ_{13} approximation $\hat{C} \approx |1 - \hat{A}|$ and

$$
\boxed{P(\nu_e \to \nu_\mu) \approx \sin^2(\theta_{23})\,\frac{\sin^2(2\theta_{13})}{(1 - \hat{A})^2}\,\sin^2\left(x(1 - \hat{A})\frac{\delta m_{31}^2}{4k}\right),} \tag{13.53}
$$

As a result of neutrino interaction with matter, both the amplitude and the frequency of oscillations are modified.

Going beyond the $\delta m_{21}^2 = 0$ approximation leads to much more complicated expressions for the different parameters which are given in the work of Martin Freund[47]. All parameters in eqs. (13.49) receive a correction proportional to $\alpha = \delta m_{21}^2/\delta m_{31}^2$. However, in a realistic and often used limit, drastic simplifications are possible. This is the case if one keeps only leading terms in α and $\sin(\theta_{13})$. In practice if one keeps, in the probability functions, only terms up to $\mathcal{O}(\alpha^2)$, $\mathcal{O}(\sin^2(\theta_{13}))$, $\mathcal{O}(\alpha\sin(\theta_{13}))$, the only correction to the parameters in eqs. (13.49) to take into account is a modification of θ_{12} to θ_{12}^m. To derive it, we turn back to eq. (13.41) and consider, assuming now $\theta_{13} \approx 0, U_{13} \approx 1$, the diagonalisation by the matrix U_{12}^m of

$$U_{12}^m \left[U_{12} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2/2k & 0 \\ 0 & 0 & \delta m_{31}^2/2k \end{pmatrix} U_{12}^\dagger + \begin{pmatrix} \sqrt{2}G_F N_e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] U_{12}^{m\dagger}. \tag{13.54}$$

This is done in sec. 13.3, the only difference being here that we define ω_1 as the largest eigenvalue and ω_2 the smallest. This amounts to exchanging ω_1 and ω_2, hence reversing the sign of the square root factor in eq (13.16). This leads to a negative θ_{12}^m, and in the large $\hat{A}_{21} = \hat{A}/\alpha \gg 1$ limit, to

$$\sin(2\theta_{12}^m) \approx -\frac{\sin(2\theta_{12})}{\hat{A}_{21}} = -\alpha\frac{\sin(2\theta_{12})}{\hat{A}}. \tag{13.55}$$

from eq. (13.17). Using this result together with eqs. (13.49) and (13.51) one reconstructs the various probability functions. All oscillatory factors now enter the formulae and, from eq. (12.29), one finds for the oscillation $\nu_e \to \nu_\mu$:

$$\begin{aligned} P(\nu_e \to \nu_\mu) \approx{}& \sin^2(\theta_{23})\frac{\sin^2(2\theta_{13})}{(1-\hat{A})^2}\sin^2\left(x(1-\hat{A})\frac{\delta m_{31}^2}{4k}\right) + \alpha^2\cos^2(\theta_{23})\frac{\sin^2(2\theta_{12})}{\hat{A}^2}\sin^2\left(x\hat{A}\frac{\delta m_{31}^2}{4k}\right) \\ &+ \alpha\,\frac{8J\cos(\delta)}{\hat{A}(1-\hat{A})}\cos\left(x\frac{\delta m_{31}^2}{4k}\right)\sin\left(x\hat{A}\frac{\delta m_{31}^2}{4k}\right)\sin\left(x(1-\hat{A})\frac{\delta m_{31}^2}{4k}\right) \\ &+ \alpha\,\frac{8J\sin(\delta)}{\hat{A}(1-\hat{A})}\sin\left(x\frac{\delta m_{31}^2}{4k}\right)\sin\left(x\hat{A}\frac{\delta m_{31}^2}{4k}\right)\sin\left(x(1-\hat{A})\frac{\delta m_{31}^2}{4k}\right). \end{aligned} \tag{13.56}$$

To obtain the terms in $\sin(\delta)$ and $\cos(\delta)$ we use respectively eqs. (12.37) and (12.39) with J as defined in eq. (12.27). We recall this expression is valid in the small $\delta m_{21}^2/\delta m_{31}^2$ and $\sin(\theta_{13})$ approximation. The effect of matter is contained in $\hat{A} = 2\sqrt{2}G_F N_e k/\delta m_{31}^2$ which changes the relative weights of the terms compared to vacuum and the magnitude of the change is energy dependent since $\hat{A} \propto k$. Taking $\alpha = 0$ one recovers a previously derived result but it is not allowed in this expression to make $\hat{A} = 0$, the vacuum limit, since the derivation was done assuming $\hat{A} = N_e \cos(2\theta_{13}/N_{\text{Res}}^{31} > \alpha$. With the present value of δm_{21}^2 this condition is, for neutrinos traversing the earth, $E_\nu > .3$ GeV. The results above thus do not apply to solar neutrinos but it does apply to atmospheric and accelerator neutrinos.

The time reversed probability $P(\nu_\mu \to \nu_e)$ is obtained from the above equation by reversing the sign of δ while for $P(\overline{\nu}_e \to \overline{\nu}_\mu)$ one reverses both the sign of $\hat{A}$ and δ. From eq. (12.33) and the above results one can obtain the oscillation probability $P(\nu_\mu \to \nu_\tau)$ in matter which are stronger than $\nu_\mu \to \nu_e$, the dominant term being proportional to $\sin^2(2\theta_{23})$ rather than $\sin^2(2\theta_{13})$.

• Discussion and order of magnitude of the parameters

We summarize here for later use the value of the parameters and the order of magnitude of the oscillating factors. One of the experimentally unsolved question is the mass ordering, *i.e.* is δm^2_{31} positive or negative ? Although the derivation above was done assuming this quantity positive it also holds with $\delta m^2_{31} < 0$ keeping $\delta m^2_{21} > 0$. In that case, $\hat{A}$ is also negative but the combinations $\alpha/\hat{A}$ and $\hat{A}\,\delta m^2_{31}$ remain positive. Similarly to the oscillations in vacuum the difference between the two hypothesis is the sign of the $\cos\delta$ term but this term is very small if $\delta \approx 3\pi/2$ (see eq. (13.56)). In matter however, since the magnitude of the oscillation depends on $\hat{A}$ one can use the energy as a parameter to probe the hierarchy hypothesis. For example, all terms with a normalisation factor in $1/(1-\hat{A})$ will be sensitive to the sign of δm^2_{31} provided of course that the associated oscillating factor $x(1-\hat{A})\delta m^2_{31}/4k$ be large enough so as not to compensate the normalisation otherwise one can expand $\sin(x(1-\hat{A})\delta m^2_{31}/4k) \approx (1-\hat{A})\sin(x\delta m^2_{31}/4k)$ and then get back the vacuum oscillation result.

$\lvert\alpha\rvert$	$\hat{A}$	$x\delta m^2_{21}/4k$	$x\delta m^2_{31}/4k$	$x\hat{A}\delta m^2_{31}/4k$
$3.\ 10^{-2}$	$.125\,(k/\mathrm{GeV})$	$10^{-4}\,(x/\mathrm{km})(k/\mathrm{GeV})^{-1}$	$3.2\ 10^{-3}\,(x/\mathrm{km})(k/\mathrm{GeV})^{-1}$	$4\ 10^{-4}\,(x/\mathrm{km})$

Table 2: *Value of the parameters controling the neutrino oscillations in the earth mantle:* $\lvert\alpha\rvert = \delta m^2_{21}/\lvert\delta m^2_{31}\rvert$, $\hat{A} = 2\sqrt{2}G_F k N_e/\lvert\delta m^2_{31}\rvert$ *with* $N_e = 1.25\ 10^{24}\ cm^{-3}$, δm^2_{21} *is positive and* $\delta m^2_{32} \approx \delta m^2_{31}$ *is assumed. The value of the masses are taken from eq. (12.22).*