Appendices

A Properties of γ^μ matrices

In Dirac representation γ^μ matrices are defined by:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix}, \tag{A.1}$$

where the τ^i are the Pauli matrices:

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.2}$$

The Pauli matrices are hermitian and they satisfy:

$$\left[\frac{\tau_i}{2}, \frac{\tau_j}{2}\right] = i\ \epsilon_{ijk}\ \frac{\tau_k}{2}, \qquad \mathrm{Tr}(\tau_i \tau_j) = 2\ \delta_{ij} \tag{A.3}$$

The matrices γ^μ have the following properties:

$$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu,\ \gamma^{0^2} = \mathbb{1}_4,\ \gamma^{i^2} = -\mathbb{1}_4,\ \sum_\mu \gamma_\mu \gamma^\mu = 4\mathbb{1}_4, \qquad \mu = 0, 1, 2, 3. \tag{A.4}$$

They satify anticommutation relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2\ g^{\mu\nu}\, \mathbb{1}_4. \tag{A.5}$$

The matrix γ_5 is defined by:

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{A.6}$$

It anticommutes with γ^μ matrices:

$$\{\gamma^5, \gamma^\nu\} = 0, \qquad \mu = 0, 1, 2, 3. \tag{A.7}$$

One proves easily:

- $\gamma_\mu \gamma_\alpha \gamma^\mu = -2\ \gamma_\alpha$
- $\gamma_\mu \gamma_\alpha \gamma_\beta \gamma^\mu = 4\ g_{\alpha\beta}\ \mathbb{1}_4$ (A.8)
- $\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\delta \gamma^\mu = -2\ \gamma_\delta \gamma_\beta \gamma_\alpha.$

For the evaluation of traces of products $\gamma^\alpha\gamma^\beta$... one has the following relations :

$$\begin{aligned}
\mathrm{Tr}(\gamma^\alpha\gamma^\beta) &= 4\, g^{\alpha\beta} \\
\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\lambda) &= 4\,[g^{\alpha\beta}\ g^{\delta\lambda} - g^{\alpha\delta}\ g^{\beta\lambda} + g^{\alpha\lambda}\ g^{\beta\delta}] \\
\mathrm{Tr}(\gamma^\alpha\gamma^\beta...) &= 0 \quad \text{for an odd number of matrices} \\
\mathrm{Tr}(\gamma^5\gamma^\alpha\gamma^\beta...) &= 0 \quad \text{for an odd number of } \gamma^\alpha \text{ matrices} \\
\mathrm{Tr}(\gamma^5\gamma^\alpha\gamma^\beta) &= 0 \\
\mathrm{Tr}(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\lambda) &= -4i\epsilon^{\alpha\beta\delta\lambda},
\end{aligned} \tag{A.9}$$

where $\epsilon^{\alpha\beta\delta\lambda}$ is the totally antisymmetric tensor under permutation of its indices with $\epsilon^{0123} = +1$. One has $\epsilon_{\alpha\beta\delta\lambda} = -\epsilon^{\alpha\beta\delta\lambda}$ and in particular $\epsilon_{0123} = -1$. A useful relation is:

$$\epsilon_{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\alpha\beta} = -2(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\mu) \tag{A.10}$$

There exists other representations due to Weyl and to Majorana which satisfy the relations eq. (A.5) to eq. (A.9). In general, when doing calculations, the explicit form of γ^μ matrices is not necessary.