B Charge conjugation $\mathcal{C}$, space reflection $\mathcal{P}$, time reversal $\mathcal{T}$

B.1 Charge conjugation $\mathcal{C}$

A fermion (electron) of charge e obeys the Dirac equation

$$((i\partial_\mu - eA_\mu)\,\gamma^\mu - m)\psi = 0. \tag{B.1}$$

Looking for a plane wave solution of type eq. (2.6) we obtain

$$\begin{aligned}
(\not{p} - e\not{A} - m)u_\alpha(p) &= 0, \quad \text{for positive energy solutions,} \\
(\not{p} + e\not{A} + m)v_\alpha(p) &= 0, \quad \text{for negative energy solutions,}
\end{aligned}$$

which suggests to interpret the negative energy solution as a positive energy one with charge $-e$, *i.e.* the antiparticle (positron). The wave function of the positron should thus satisfy the same equation as the electron with an opposite charge

$$((i\partial_\mu + eA_\mu)\,\gamma^\mu - m)\psi^c = 0. \tag{B.2}$$

The solution ψ^c can be constructed in the following way. From the first equation above one has

$$(-(i\partial_\mu + eA_\mu)\,\gamma^{\mu^*} - m)\psi^* = 0. \tag{B.3}$$

We look for ψ^c under the form

$$\boxed{\psi^c = \mathcal{C}\gamma_0\psi^*,} \tag{B.4}$$

where $\mathcal{C}$ is a 4×4 matrix. Then eq. (B.3) yields after multiplication on the left by $\mathcal{C}\gamma^0$:

$$(\mathcal{C}\gamma^0)(-(i\partial_\mu + eA_\mu)\,\gamma^{\mu^*} - m)\,(\mathcal{C}\gamma^0)^{-1}\psi^c = 0, \tag{B.5}$$

and, if one finds a matrix $\mathcal{C}$ such that:

$$(\mathcal{C}\gamma^0)\,\gamma^{\mu^*} = -\gamma^\mu\,(\mathcal{C}\gamma^0), \tag{B.6}$$

then we recover eq. (B.2). In our representation of γ_μ matrices, we have

$$\gamma^{\mu^*} = \gamma^\mu,\ \mu = 0,1,3; \qquad \gamma^{\mu^*} = -\gamma^\mu,\ \mu = 2, \tag{B.7}$$

so that the choice of the real matrix

$$\boxed{(\mathcal{C}\gamma^0) = i\gamma^2} \quad \Leftrightarrow \quad \boxed{\mathcal{C} = i\gamma^2\gamma^0} \tag{B.8}$$

satisfy the condition (B.6) which is equivalent to:

$$\gamma_2 \gamma_\mu \gamma_2 = \gamma_\mu^*. \tag{B.9}$$

Using the relations eqs. (A.4) and (A.5), it is easy to prove

$$\mathcal{C} = -\mathcal{C}^{-1} = -\mathcal{C}^\dagger = -\mathcal{C}^T \quad \text{and} \quad \mathcal{C}\gamma^5\mathcal{C}^{-1} = \gamma^5 \;\Leftrightarrow\; \gamma_2\gamma_5\gamma_2 = \gamma_5. \tag{B.10}$$

Under charge conjugation, the wave function ψ which satisfies eq. (B.1) becomes

$$\boxed{\psi^c = \mathcal{C}\gamma^0\psi^* = \mathcal{C}\overline{\psi}^T = i\gamma^2\psi^* \qquad \Leftrightarrow \qquad \overline{\psi^c} = i\psi^T\gamma_2\gamma_0,} \tag{B.11}$$

solution of eq. (B.2).

Let us first discuss free massless chiral spinors, eqs. (3.28) and (3.30), important in the construction of the Standard Model. The application of $\mathcal{C}$ parity yields:

$$\begin{aligned}
(u_L)^c(p) &= \mathcal{C}\gamma_0\; u_L^*(p) = i\gamma_2\; u_L^*(p) = \sqrt{\omega}\; i\gamma_2 \begin{pmatrix} \chi_L^* \\ -\chi_L^* \end{pmatrix} = -\sqrt{\omega}\begin{pmatrix} \chi_R \\ \chi_R \end{pmatrix} = v_L(p) \\
(u_R)^c(p) &= \mathcal{C}\;\gamma_0\; u_R^*(p) = i\gamma_2\; u_R^*(p) = \sqrt{\omega}\; i\gamma_2 \begin{pmatrix} \chi_R^* \\ \chi_R^* \end{pmatrix} = \sqrt{\omega}\begin{pmatrix} -\chi_L \\ \chi_L \end{pmatrix} = v_R(p),
\end{aligned} \tag{B.12}$$

and thus, the $\mathcal{C}$ operator transforms the wave-function of a positive energy spinor (electron) into the wave-function of a negative energy one (positron) of the same helicity (similar relations exist for $(v_L)^c$ and $(v_R)^c$). Recalling the definition of $\psi_L(x)$, eq. (3.23),

$$\psi_L(x) = \int \frac{d^3p}{(2\pi)^3 2\omega}\left[b_L(p)\; u_L(p)\; e^{-ip.x} + d_R^\dagger(p)\; v_R(p)\; e^{ip.x}\right]$$

its $\mathcal{C}$ transformed is:

$$(\psi_L)^c(x) = \int \frac{d^3p}{(2\pi)^3 2\omega}\left[d_R(p)\; u_R(p)\; e^{-ip.x} + b_L^\dagger(p)\; v_L(p)\; e^{ip.x}\right], \tag{B.13}$$

which destroys a right-handed antifermion with wave-function $u_R(p)$ and creates a left-handed fermion with $v_L(p)$. Equivalently, in a compact form, if one writes $\psi_L = (1-\gamma^5)\psi/2$, its charge conjugate is:

$$(\psi_L)^c = i\gamma^2\;\psi_L^* = \frac{1+\gamma^5}{2}\, i\gamma^2\psi^* = \frac{1+\gamma^5}{2}\,\psi^c = (\psi^c)_R, \tag{B.14}$$

a right-handed wave-function. Likewise the $\mathcal{C}$ conjugate of a right-handed wave-function is left-handed

$$(\psi_R)^c = \frac{1-\gamma^5}{2}\,\psi^c = (\psi^c)_L. \tag{B.15}$$

Going back to the general case, it is easy to show that under $\mathcal{C}$ parity the helicity projection operators satisfy:

$$i\gamma_2\,\Sigma^{\pm *}(s) = i\gamma_2\,\frac{(1 \pm \gamma_5\,\not{s}^*)}{2} = \frac{(1 \pm \gamma_5\,\not{s})}{2}\,i\gamma_2 = \Sigma^{\pm}(s)\,i\gamma_2,$$

and the energy projection operators satisfy:

$$i\gamma_2\,\Lambda^{\pm *}(p) = i\gamma_2\,\frac{\pm\not{p}^* + m}{2} = \frac{\mp\not{p} + m}{2}\,i\gamma_2 = \Lambda^{\mp}(p)\,i\gamma_2,$$

where one has used the relations eq. (B.7). Thus, a solution of the Dirac equation of positive (resp. negative) energy and given helicity becomes a solution of negative (resp. positive) energy of the same helicity:

$$i\gamma_2\,\Sigma^{\pm *}(s)\,\Lambda^{\pm *}(p)\,\psi^*(p,x)) \;=\; \Sigma^{\pm}(s)\,\Lambda^{\mp}(p)\,\psi^c(p,x), \tag{B.16}$$

It is useful to list the transformation of fermion bilinears under $\mathcal{C}$. They easily derived from eqs.(B.9) to (B.11), remembering the $-$ sign (due to Fermi statistics) when transposing the expressions to obtain the right hand-side, and one finds:

$$\begin{aligned}
\overline{\psi_2^c}(x)\psi_1^c(x) &= \overline{\psi}_1(x)\psi_2(x),\\
\overline{\psi_2^c}(x)\gamma^5\psi_1^c(x) &= \overline{\psi}_1(x)\gamma^5\psi_2(x),\\
\overline{\psi_2^c}(x)\gamma^\nu\psi_1^c(x) &= -\overline{\psi}_1(x)\gamma^\nu\psi_2(x),\\
\overline{\psi_2^c}(x)\gamma^\nu\gamma^5\psi_1^c(x) &= \overline{\psi}_1(x)\gamma^\nu\gamma^5\psi_2(x).
\end{aligned} \tag{B.17}$$

B.2 Space reflection $\mathcal{P}$

The space reflection, or parity transformation is defined by :

$$x_0 \to x_0' = x_0, \qquad \mathbf{x} \to \mathbf{x}' = -\mathbf{x}. \tag{B.18}$$

The transformation is parameterised in the following way

$$x'^\nu = a^\nu_\mu x^\mu \quad \text{with} \quad a^\nu_\mu = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{B.19}$$

Knowing $\psi(x_0,\mathbf{x})$ satisfying the free Dirac equation, we look for the form of the solution obtained under a space reflection. We write

$$\psi'(x_0,\mathbf{x}') = \psi'(x_0,-\mathbf{x}) = \mathcal{P}\psi(x_0,\mathbf{x}), \qquad \text{thus} \qquad \psi(x_0,\mathbf{x}) = \mathcal{P}^{-1}\psi'(x_0,-\mathbf{x}) \tag{B.20}$$

From the free Dirac equation

$$(i\frac{\partial}{\partial x^\mu}\gamma^\mu - m)\psi(x_0, \mathbf{x}) = 0, \tag{B.21}$$

we obtain, using

$$\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\mu} = \frac{\partial}{\partial x'^\nu}a^\nu_\mu, \tag{B.22}$$

$$(i\frac{\partial}{\partial x^\mu}\gamma^\mu - m)\mathcal{P}^{-1}\psi'(x_0, \mathbf{x}') = (i\frac{\partial}{\partial x'^\nu}a^\nu_\mu\gamma^\mu - m)\mathcal{P}^{-1}\psi'(x_0, \mathbf{x}') = 0, \tag{B.23}$$

which leads to

$$(i\frac{\partial}{\partial x'^\nu}\ a^\nu_\mu\ \mathcal{P}\gamma^\mu\mathcal{P}^{-1} - m)\psi'(x_0, \mathbf{x}') = 0. \tag{B.24}$$

If we find a matrix $\mathcal{P}$ such that $a^\nu_\mu\ \mathcal{P}\gamma^\mu = \gamma^\nu\mathcal{P}$, then $\psi'(x_0, \mathbf{x}')$ will be a solution. Such a matrix should commute with γ^0 and anticommute with $\vec{\gamma}$. Obviously

$$\boxed{\mathcal{P} = e^{i\phi}\gamma^0} \tag{B.25}$$

has such a property. Thus

$$\psi'(x_0, \mathbf{x}') = e^{i\phi}\gamma^0\psi(x_0, \mathbf{x}), \qquad \bar{\psi}'(x_0, \mathbf{x}') = e^{-i\phi}\psi^\dagger(x_0, \mathbf{x}). \tag{B.26}$$

Note that the parity operator reverses the fermion helicity. Thus a massless left-handed fermion becomes right-handed:

$$\begin{aligned} \mathcal{P}\psi_L(x_0, \mathbf{x}) &= \gamma_0\,\frac{1-\gamma_5}{2}\,\psi(x_0, \mathbf{x}) \\ &= \frac{1+\gamma_5}{2}\,\psi'(x_0, \mathbf{x}') \ = \ \psi'_R(x_0, \mathbf{x}') \end{aligned} \tag{B.27}$$

(where for simplicity we ignore an irrelevant phase). In terms of Dirac spinors one has:

$$\gamma_0\,u(\boldsymbol{p}) = u(-\boldsymbol{p}), \qquad \gamma_0\,v(\boldsymbol{p}) = -v(-\boldsymbol{p}), \tag{B.28}$$

as can be immediatly verified from eqs (3.12). It is easy to check the behavior of the fermion bilinears under a parity transformation:

$$\begin{aligned} \overline{\psi'_2}(x_0, \mathbf{x}')\psi'_1(x_0, \mathbf{x}') &= \overline{\psi}_2(x_0, \mathbf{x})\psi_1(x_0, \mathbf{x}), \quad \text{a scalar} \\ \overline{\psi'_2}(x_0, \mathbf{x}')\gamma^5\psi'_1(x_0, \mathbf{x}') &= -\overline{\psi}_2(x_0, \mathbf{x})\gamma^5\psi_1(x_0, \mathbf{x}), \quad \text{a pseudoscalar} \\ \overline{\psi'_2}(x_0, \mathbf{x}')\gamma^\nu\psi'_1(x_0, \mathbf{x}') &= a^\nu_\mu\overline{\psi}_2(x_0, \mathbf{x})\gamma^\mu\psi_1(x_0, \mathbf{x}), \quad \text{a vector} \\ \overline{\psi'_2}(x_0, \mathbf{x}')\gamma^\nu\gamma^5\psi'_1(x_0, \mathbf{x}') &= -a^\nu_\mu\overline{\psi}_2(x_0, \mathbf{x})\gamma^\mu\gamma^5\psi_1(x_0, \mathbf{x}), \quad \text{a pseudovector or axial vector} \end{aligned} \tag{B.29}$$

B.3 Variance and invariance of the lagrangien under $\mathcal{C}$ and $\mathcal{CP}$

From the above discussion, it is easy to obtain the transformation properties of the lagrangien. The easiest case is that of QED:

$$\mathcal{L}_{QED} = \overline{\psi}(x)(i\partial\!\!\!/ - eA\!\!\!/(x) - m)\psi(x). \tag{B.30}$$

Under $\mathcal{P}$ all vectors such as $x'_\mu, \partial'_\mu,\ A'_\mu(x')$ transform as $a^\nu_\mu x_\nu, a^\nu_\mu \partial_\nu, a^\nu_\mu\ A_\nu(x)$ and $\bar{\psi}'(x_0, \mathbf{x}')\gamma^\mu\psi'(x_0, \mathbf{x}')$ $\to\ a^\mu_\nu \bar{\psi}(x_0, \mathbf{x})\gamma^\nu\psi(x_0, \mathbf{x})$ so that

$$\overline{\psi'}(x')(i\partial\!\!\!/' - eA\!\!\!/'(x') - m)\psi'(x') \tag{B.31}$$

reduces to the lagrangien above. The transformation is also very simple under $\mathcal{C}$. The $U(1)$ gauge transformation, $\psi'(x) \to \exp(-ie\alpha(x))\psi(x)$, implies $\psi^{c\prime}(x) \to \exp(ie\alpha(x))\psi^c(x)$ and the $U(1)$ gauge invariance of $\mathcal{L}_{QED}$ leads to $A^c_\mu(x) = -A_\mu(x)$ (use eqs. (B.17) to prove the invariance). For the derivative term it is a bit more tricky since

$$\begin{aligned}
\overline{\psi^c}(x)\ i\partial\!\!\!/\ \psi^c(x) &= \overline{\psi^c}(x)\ i\gamma^\mu\ \overrightarrow{\partial_\mu}\psi^c(x)\ =\ \psi^T(x)\gamma_0\ i\gamma^{\mu*}\ \overrightarrow{\partial_\mu}\psi^*(x) \\
&= -\psi^\dagger(x)\overleftarrow{\partial_\mu}\ i\gamma^{\mu\dagger}\gamma_0\ \psi(x)\ =\ -\psi^\dagger(x)\overleftarrow{\partial_\mu}\ \gamma_0 i\gamma^\mu\ \psi(x) \\
&= \overline{\psi}(x)\ i\gamma^\mu\overrightarrow{\partial_\mu}\ \psi(x)\ =\ \overline{\psi}(x)\ i\partial\!\!\!/\ \psi(x).
\end{aligned} \tag{B.32}$$

One goes from the first to the second line by transposing the expression keeping in mind the - sign for the anticommutation of the fermions and from the second line to the last one by a partial integration neglecting, as usual, a total derivative. This proves the invariance of the QED lagrangian under $\mathcal{C}$, $\mathcal{P}$ and therefore $\mathcal{CP}$ transformations.

On the contrary a theory with an interaction term of the form $\overline{\psi}(x)\gamma^\mu(1-\gamma_5)\psi(x)$ is not invariant under $\mathcal{C}$ or $\mathcal{P}$ since this term becomes, up to an overall sign, $\overline{\psi}(x)\gamma^\mu(1+\gamma_5)\psi(x)$ (use eqs. (B.17) and (B.29)), and one can say that there is maximum violation of these symmetries. However it is invariant under $\mathcal{CP}$. The case of the Standard Model with three generations is a bit more subtle. Consider the charged current piece eq. (11.9) written in the mass eigenstate basis. Denoting $\mathbf{V}$ the **CKM** matrix, with $\mathbf{V}_{ij} = v_{ij}, \mathbf{V}^\dagger{}_{ji} = v^*_{ij}$, the charged current is written:

$$\begin{aligned}
\mathcal{L}_F(\text{charged current}) &= \frac{e}{\sqrt{2}\sin\theta_W}\,[\overline{\mathbf{u}}_L W^*_\mu\gamma^\mu\mathbf{V}\ \mathbf{d}_L\ +\ \overline{\mathbf{d}}_L\mathbf{V}^\dagger W_\mu\gamma^\mu\mathbf{u}_L\ ] \\
&= \frac{e}{\sqrt{2}\sin\theta_W}\,[v_{ij}\overline{u}^j_L W^*_\mu\gamma^\mu\ d^j_L + v^*_{ij}\overline{d}^j_L W_\mu\gamma^\mu\ u^i_L] \\
&= \frac{e}{2\sqrt{2}\sin\theta_W}\,[v_{ij}\overline{u}^i\ W^*_\mu\gamma^\mu\ (1-\gamma_5)\ d^j + v^*_{ij}\overline{d}^j\ W_\mu\gamma^\mu(1-\gamma_5)\ u^i],
\end{aligned} \tag{B.33}$$

where the index i, j run over the number of fermion generations. If $\boldsymbol{\psi}$ is in the fundamental representation of the unitary group $\mathbf{G}$, with generators τ^a the generators operating on the $\boldsymbol{\psi^c}$ fields are τ^{a*} and the conjugate of the gauge boson is $\boldsymbol{W}_\mu^c = -W_\mu^a \tau^{a*} = -\boldsymbol{W}_\mu^*$, so that $W_\mu \leftrightarrow -W_\mu^*$ under $\mathcal{C}$ parity (with the definition of W_μ given after eq. (5.43)). Under charge conjugation, $\mathcal{L}_F$ becomes:

$$\begin{aligned}\mathcal{L}_F^{\mathcal{C}}(\text{charged current}) &= \frac{e}{2\sqrt{2}\sin\theta_W}\,[v_{ij}\overline{u^c}^i\; W_\mu^{c*}\gamma^\mu\;(1-\gamma_5)\; d^{cj} + v_{ij}^*\overline{d^c}^j\; W_\mu^c\gamma^\mu(1-\gamma_5)\; u^{ci}] \\ &= \frac{e}{2\sqrt{2}\sin\theta_W}\,[v_{ij}\overline{d}^j\; W_\mu\gamma^\mu\;(1+\gamma_5)\; u^i + v_{ij}^*\overline{u}^i\; W_\mu^*\gamma^{\mu*}(1+\gamma_5)\; d^j], \end{aligned} \tag{B.34}$$

where we have used eqs. (B.17). If we do furthermore a $\mathcal{P}$ transformation on this expression we obtain:

$$\mathcal{L}_F^{\mathcal{CP}}(\text{charged current}) = \frac{e}{2\sqrt{2}\sin\theta_W}\,[v_{ij}\overline{d}^j\; W_\mu\gamma^\mu\;(1-\gamma_5)\; u^i + v_{ij}^*\overline{u}^i\; W_\mu^*\gamma^{\mu*}(1-\gamma_5)\; d^j], \tag{B.35}$$

since, following eqs. (B.29), the term in $W_\mu\gamma^\mu$ is invariant while $W_\mu\gamma^\mu\gamma_5$ changes sign. This is identical to eq. (B.33) except for the $v_{ij} \leftrightarrow v_{ij}^*$ factors interchanged between the two terms of the expression: if the **CKM** matrix were real then the lagrangian would be invariant under $\mathcal{CP}$, in other words the phase of the **CKM** matrix is at the origin of $\mathcal{CP}$ violation in the Standard Model since all other terms in the lagrangien are invariant under $\mathcal{CP}$.

B.4 Time reflection $\mathcal{T}$

The time-reflection transformation takes the coordinate $x = (x_0, \mathbf{x})$ to $x' = (-x_0, \mathbf{x})$. This transformation can be written

$$x'^\nu = a_\mu^\nu x^\mu \quad \text{with} \quad a_\mu^\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{B.36}$$

Denoting $\psi'(x') \equiv \psi'(-x_0, \mathbf{x})$ the free time-reflected wave function, we attempt to construct it under the form

$$\psi'(x') = \mathcal{T}\psi^*(x), \qquad \Rightarrow \qquad \psi^*(x) = \mathcal{T}^{-1}\psi'(x'), \tag{B.37}$$

where $\mathcal{T}$ is a 4×4 constant matrix and $\psi(x)$ is the solution of the free Dirac equation with

$$(-i\frac{\partial}{\partial x_\mu}\gamma^{\mu*} - m)\psi^*(x) = 0, \qquad \Rightarrow \qquad (-i\frac{\partial}{\partial x'_\nu}a_\mu^\nu\gamma^{\mu*} - m)\mathcal{T}^{-1}\psi(x') = 0. \tag{B.38}$$

Multiplying to the left by $\mathcal{T}$ we obtain

$$(-i\frac{\partial}{\partial x'_\nu}a_\mu^\nu\mathcal{T}\gamma^{\mu*}\mathcal{T}^{-1} - m)\psi'(x') = 0, \tag{B.39}$$

and $\psi'(x')$ will be a solution of the Dirac equation, *i.e.* will satisfy

$$(i\frac{\partial}{\partial x'_\nu}\gamma^\nu - m)\psi'(x') = 0, \tag{B.40}$$

if we find a matrix such that

$$a^\nu_\mu \mathcal{T}\gamma^{\mu *}\mathcal{T}^{-1} = -\gamma^\nu. \tag{B.41}$$

Recalling that $\gamma^{\mu *} = \gamma^\mu$ for $\mu = 0, 1, 3$ and $\gamma^{2*} = -\gamma^2$, the above conditions reduce to $\mathcal{T}\gamma^i = \gamma^i\mathcal{T}$ for $i = 0, 2$ and $\mathcal{T}\gamma^j = -\gamma^j\mathcal{T},\ j = 1, 3$. The matrix

$$\mathcal{T} = i\gamma^1\gamma^3 \tag{B.42}$$

satisfies the required conditions and we have thus

$$\boxed{\psi'(-x_0, \mathbf{x}) = i\gamma^1\gamma^3\psi^*(x_0, \mathbf{x})} \tag{B.43}$$

the solution for the free wave function evolving backward in time. For an interacting fermion in QED, under time reversal the potential $A'^\mu(x')$ is related to $A^\mu(x)$ by $A'^0(x') = A^0(x),\ A^i(x') = -A^i(x)$, since the current reverses sign when the arrow of time is reversed and under this condition we can show that QED is invariant under time reversal.

Combining the symmetries $\mathcal{P}$ and $\mathcal{T}$ one can construct the wave function of an electron evolving backward in space-time,

$$\begin{aligned} \psi^{\mathcal{PT}}(-x) &= \mathcal{PT}\psi(x) = \gamma^0[i\gamma^1\gamma^3\psi^*(x)] \\ &= \gamma^5 i\gamma^2\psi^*(x) \\ &= \gamma_5\psi^c(x), \end{aligned} \tag{B.44}$$

where we introduced the wave-function of the positron via eq. (B.11), Thus the wave function of a (right-handed) positron is that of a (left-handed) electron moving backward in space-time (up to an irrelevant phase factor).