3 Fermions, chirality, helicity

3.1 Fermions : chirality

We saw that the Fermi model involves charged transitions such as $\bar{\psi}_d\gamma_\mu(1-\gamma_5)\psi_u$ ou $\bar{\psi}_e\gamma_\mu(1-\gamma_5)\psi_{\nu_e}$, *i.e.* charged currents of a particular type : the fermion interacts only through the combination $(1-\gamma_5)\psi$. We can always write :

$$\psi = \psi_- + \psi_+, \qquad \text{with} \qquad \psi_- = \frac{1-\gamma_5}{2}\psi,\ \psi_+ = \frac{1+\gamma_5}{2}\psi \tag{3.1}$$

The spinors ψ_- and ψ_+ have a definite *chirality* defined by their transformation when applying γ_5 :

$$\gamma_5\ \psi_- = -\psi_-, \qquad \gamma_5\ \psi_+ = \psi_+. \tag{3.2}$$

ψ_-, ψ_+ have *negative, positive chirality* respectively. The combinations

$$P^- = \frac{1-\gamma_5}{2}, \qquad P^+ = \frac{1+\gamma_5}{2} \tag{3.3}$$

are projection operators satisfying:

$$P^+ + P^- = 1, \qquad P^+P^- = 0, \qquad (P^+)^2 = P^+, \qquad (P^-)^2 = P^-. \tag{3.4}$$

Only negative chirality fermions are sensitive to the weak interactions. It is useful to note that:

$$\overline{\psi_-} = \overline{\psi}\,\frac{1+\gamma_5}{2}, \qquad \overline{\psi_+} = \overline{\psi}\,\frac{1-\gamma_5}{2}. \tag{3.5}$$

3.2 Fermions : positive and negative energy solutions

When using the plane wave decomposition of the spinor, eq. (2.6), the free Dirac equation $(i\partial\!\!\!/-m)\psi = 0$ implies:

$$(p\!\!\!/ - m)\ u_\alpha(p) = 0, \qquad\qquad (p\!\!\!/ + m)\ v_\alpha(p) = 0 \tag{3.6}$$

on the positive ($u_\alpha \exp(-ipx)$, see eq. (2.6)) and negative ($v_\alpha \exp(ipx)$) energy component respectively. At rest, $\mathbf{p} = \mathbf{0}$, and using the Dirac representation of γ_μ matrices given in appendix, they reduce to:

$$\begin{aligned} m(\gamma^0-1)u_\alpha \qquad &\Rightarrow \qquad \begin{pmatrix} 0 & 0 \\ 0 & -2\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} \chi_\alpha \\ 0 \end{pmatrix} = 0 \\ m(\gamma^0+1)v_\alpha \qquad &\Rightarrow \qquad \begin{pmatrix} 2\mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \chi_\alpha \end{pmatrix} = 0 \end{aligned} \tag{3.7}$$

where we have introduced the 2-component spinors :

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad u_\alpha = \begin{pmatrix} \chi_\alpha \\ 0 \end{pmatrix}, \quad v_\alpha = \begin{pmatrix} 0 \\ \chi_\alpha \end{pmatrix}, \qquad \alpha = 1, 2. \tag{3.8}$$

Since one has $\tau^3\chi_1 = \chi_1,\ \tau^3\chi_2 = -\chi_2$ one says that χ_1 has spin up and χ_2 spin down and[6]

$$\frac{1+\tau^3}{2} \quad \text{and} \quad \frac{1-\tau^3}{2} \tag{3.9}$$

are respectively the spin up and spin down projection operators for the 2-component spinors. When $\mathbf{p} \neq \mathbf{0}$, to obtain the spinors $u_\alpha(p)$ and $v_\alpha(p)$ one can apply a Lorentz boost to the solution in the rest frame or, more simply, observe that:

$$\boxed{u_\alpha(p) = \frac{1}{\sqrt{\omega+m}}(\not{p}+m)u_\alpha, \qquad v_\alpha(p) = \frac{1}{\sqrt{\omega+m}}(-\not{p}+m)v_\alpha,} \tag{3.10}$$

satisfy eqs. (3.6) respectively. The factor $1/\sqrt{\omega+m}$ is the chosen normalisation factor such that:

$$\begin{aligned} \bar{u}_\alpha(p)\, u_\beta(p) &= 2m\ \delta_{\alpha\beta}, & u^\dagger_\alpha(p)\, u_\beta(p) &= 2\omega\ \delta_{\alpha\beta}, \\ \bar{v}_\alpha(p)\, v_\beta(p) &= -2m\ \delta_{\alpha\beta}, & v^\dagger_\alpha(p)\, v_\beta(p) &= 2\omega\ \delta_{\alpha\beta}. \end{aligned} \tag{3.11}$$

Explicitely, one has in terms of two component spinors:

$$\boxed{u_\alpha(p) = \frac{1}{\sqrt{\omega+m}}\begin{pmatrix} (\omega+m)\,\chi_\alpha \\ \mathbf{p}.\boldsymbol{\tau}\ \chi_\alpha \end{pmatrix}, \qquad v_\alpha(p) = \frac{1}{\sqrt{\omega+m}}\begin{pmatrix} \mathbf{p}.\boldsymbol{\tau}\ \chi_\alpha \\ (\omega+m)\,\chi_\alpha \end{pmatrix}.} \tag{3.12}$$

The solution $u_\alpha(p)$ is the positive energy spinor while $v_\alpha(p)$ is called the negative energy one with momentum $(-\omega, -\mathbf{p})$. In particular, for a boost of magnitude η in the z direction, the positive energy spinors have momentum $p = (\omega, 0, 0, p_z)$, with $\omega = m\cosh\eta,\ p_z = m\sinh\eta$, and they become:

$$u_\alpha(p) = \frac{1}{\sqrt{\omega+m}}\begin{pmatrix} (\omega+m)\chi_\alpha \\ p_z\tau^3\ \chi_\alpha \end{pmatrix} \Rightarrow u_1(p) = \frac{1}{\sqrt{\omega+m}}\begin{pmatrix} \omega+m \\ 0 \\ p_z \\ 0 \end{pmatrix},\ u_2(p) = \frac{1}{\sqrt{\omega+m}}\begin{pmatrix} 0 \\ \omega+m \\ 0 \\ -p_z \end{pmatrix}, \tag{3.13}$$

while the negative energy solutions with momentum $-p$ are:

$$v_\alpha(p) = \frac{1}{\sqrt{\omega+m}}\begin{pmatrix} p_z\ \tau^3\ \chi_\alpha \\ (\omega+m)\chi_\alpha \end{pmatrix} \Rightarrow v_1(p) = \frac{1}{\sqrt{\omega+m}}\begin{pmatrix} p_z \\ 0 \\ \omega+m \\ 0 \end{pmatrix},\ v_2(p) = \frac{1}{\sqrt{\omega+m}}\begin{pmatrix} 0 \\ -p_z \\ 0 \\ \omega+m \end{pmatrix}. \tag{3.14}$$

In general, it is useful to introduce operators which project out positive and negative energy states. They are defined by:

$$\Lambda_\pm = \frac{\pm\not{p}+m}{2m}, \tag{3.15}$$

[6]The Pauli matrices τ^i are given in appendix A.

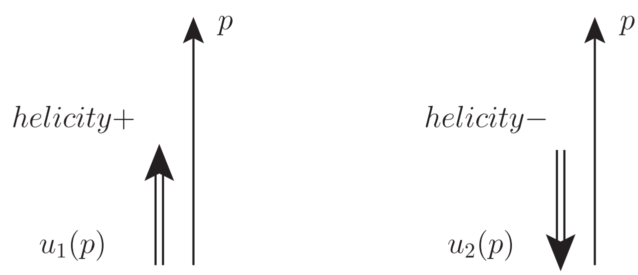
and they satisfy the required relations:

$$\Lambda_-(p) + \Lambda_+(p) = 1 \ , \quad \Lambda_-(p)\Lambda_+(p) = \Lambda_+(p)\Lambda_-(p) = 0 \ , \quad (\Lambda_-(p))^2 = \Lambda_-(p) \ , \quad (\Lambda_+(p))^2 = \Lambda_+(p) \ . \tag{3.16}$$

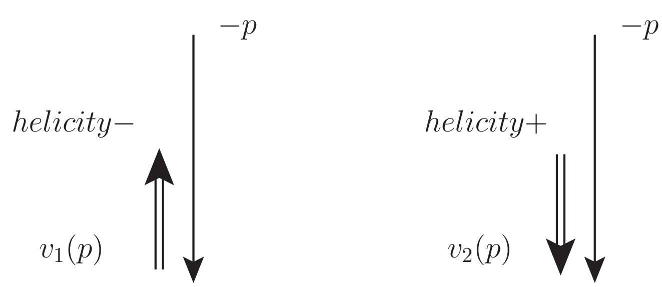
Thus $\Lambda_\pm\psi(p,x)$ respectively project the positive and negative energy solutions of $\psi(x)$ in eq. (2.6). We discuss in appendix B.1 the interpretation of the negative energy solution as a positive energy antiparticle.

3.3 Fermions : helicity

When applying a boost along the z-axis one does not change the orientation of the fermion spin, as shown in the figure below, so that the projection of the fermion spin along the momentum is positive for $u_1(p)$ (spin up) and negative for $u_2(p)$ (spin down) : one says that $u_1(p)$ has *positive helicity* or is *right-handed* and is denoted by $u_R(p)$, while $u_2(p)$ has *negative helicity* or is *left-handed* and is denoted by $u_L(p)$.



For the negative energy solutions the situation is opposite and $v_1(p) = v_L(p)$ has *negative helicity* or is *left-handed* and $v_2(p) = v_R(p)$ has *positive helicity* or is *right-handed* as shown below



For a spinor of momentum $\mathbf{p}$ one defines the helicity projection operator

$$\mathcal{S}^\pm(\hat{\mathbf{p}}) = \frac{1 \pm \mathcal{T}.\,\hat{\mathbf{p}}}{2}, \qquad \mathcal{T} = \begin{pmatrix} \boldsymbol{\tau} & 0 \\ 0 & \boldsymbol{\tau} \end{pmatrix}, \qquad \hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|} \tag{3.17}$$

with $\hat{\mathbf{p}}$ the unit vector in the direction of the momentum. Applying these operators to the positive energy spinors one finds ($(\boldsymbol{\tau}\hat{\mathbf{p}})^2 = 1$):

$$\mathcal{S}^{\pm}(\hat{\mathbf{p}})u_\alpha(p) = \frac{1}{2\sqrt{\omega+m}} \begin{pmatrix} (\omega+m)(1 \pm \boldsymbol{\tau}\hat{\mathbf{p}})\chi_\alpha \\ p(\pm 1 + \boldsymbol{\tau}\hat{\mathbf{p}})\chi_\alpha \end{pmatrix}. \tag{3.18}$$

If $\hat{\mathbf{p}}$ is in the direction of Oz, the helicity projection operators applied on the spinors reduce to

$$\mathcal{S}^{\pm}(\hat{\mathbf{p}})u_\alpha(p) = \frac{1}{2\sqrt{\omega+m}} \begin{pmatrix} (\omega+m)(1 \pm \tau_3)\chi_\alpha \\ p(\pm 1 + \tau_3)\chi_\alpha \end{pmatrix}, \tag{3.19}$$

showing that $u_1(p)$ is right-handed, and $u_2(p)$ is left-handed as found before. For negative energy spinors, since they have momentum $-\mathbf{p}$, $\mathcal{S}^+(-\hat{\mathbf{p}})$ projects out positive helicity and $\mathcal{S}^-(-\hat{\mathbf{p}})$ projects our negative helicity. The operators $S^{\pm}$ are helicity projection operators and satisfy:

$$(S^{\pm}(\mathbf{p}))^2 = S^{\pm}(\mathbf{p}), \qquad S^+(\mathbf{p})\, S^-(\mathbf{p}) = 0, \qquad S^+(\mathbf{p}) + S^-(\mathbf{p}) = \mathbb{1}_2 \tag{3.20}$$

• Massless spinors : helicity and chirality

In the Standard Model, at high energy, quarks of light flavours and neutrinos are often treated as massless. Considering massless spinors with a generic momentum p one has:

$$\boxed{u_\alpha(p) = \sqrt{\omega} \begin{pmatrix} \chi_\alpha \\ \hat{\mathbf{p}}.\boldsymbol{\tau}\ \chi_\alpha \end{pmatrix}, \qquad v_\alpha(p) = \sqrt{\omega} \begin{pmatrix} \hat{\mathbf{p}}.\boldsymbol{\tau}\ \chi_\alpha \\ \chi_\alpha \end{pmatrix}, \quad \alpha = 1 \text{ or } 2.} \tag{3.21}$$

When acting on positive energy spinors $u_\alpha(p)$, the helicity projection operator and $P^{\pm}$, the chirality projection operators of eq. (3.3), give the same result:

$$\mathcal{S}^{\pm}(\hat{\mathbf{p}})\, u_\alpha(p) = P^{\pm}\, u_\alpha(p) = \frac{\sqrt{\omega}}{2} \begin{pmatrix} (1 \pm \boldsymbol{\tau}\hat{\mathbf{p}})\chi_\alpha \\ (\pm 1 + \boldsymbol{\tau}\hat{\mathbf{p}})\chi_\alpha \end{pmatrix}, \qquad \alpha = 1, 2\,,$$

This shows that positive chirality and right-handed helicity are the same and likewise for negative chirality and left-handed helicity. For spinors $v_\alpha(p)$ one finds instead:

$$\mathcal{S}^{\pm}(-\hat{\mathbf{p}})\, v_\alpha(p) = P^{\mp}\, v_\alpha(p) = \frac{\sqrt{\omega}}{2} \begin{pmatrix} (\mp 1 + \boldsymbol{\tau}\hat{\mathbf{p}})\chi_\alpha \\ (1 \mp \boldsymbol{\tau}\hat{\mathbf{p}})\chi_\alpha \end{pmatrix}, \qquad \alpha = 1, 2\,,$$

thus a right-handed negative energy spinor has negative chirality and a left-handed one positive chirality. Thus if one constructs a massless spinor $u(p)$ as a linear combination of $u_\alpha, \alpha = 1, 2$, then $u_L(p) = \frac{(1-\gamma_5)}{2}\, u(p)$ and $u_R(p) = \frac{(1+\gamma_5)}{2}\, u(p)$ are respectively left-handed and right-handed spinors, while $v_L(p) = \frac{(1+\gamma_5)}{2}\, v(p)$ is left-handed and $v_R(p) = \frac{(1-\gamma_5)}{2}\, v(p)$ right-handed, so helicity = chirality for positive energy spinors but helicity = − chirality for negative energy ones .

To summarise, in the massless case, from the definition of $\psi(x)$ in eq. (2.6), the combination

$$\psi_L(x) = \frac{1-\gamma_5}{2}\,\psi(x) \tag{3.22}$$

- destroys a left-handed fermion, with wave function $u_L(p)$ and creates a right-handed antifermion with wave function $v_R(p)$, eqs. (3.28), (3.30),

$$\psi_L(x) = \int \frac{d^3p}{(2\pi)^3 2\omega}\left[b_L(p)\,u_L(p)\,e^{-ip.x} + d_R^\dagger(p)\,v_R(p)\,e^{ip.x}\right] \tag{3.23}$$

and *mutatis mutandis*:

$$\psi_R(x) = \frac{1+\gamma_5}{2}\,\psi(x) \tag{3.24}$$

- destroys a right-handed fermion, with wave function $u_R(p)$ and creates a left-handed antifermion with wave function $v_L(p)$.

$$\psi_R(x) = \int \frac{d^3p}{(2\pi)^3 2\omega}\left[b_R(p)\,u_R(p)\,e^{-ip.x} + d_L^\dagger(p)\,v_L(p)\,e^{ip.x}\right] \tag{3.25}$$

Thus, the Fermi interaction, discussed in the previous section, concerns only left-handed fermions and right-handed antifermions.

• Massless chiral spinors

It is easy and amusing (as well as useful for neutrino physics) to find the explicit form of massless chiral spinors of arbitrary momentum. For instance, for positive energy spinors one has, using expressions (3.12) :

$$\begin{aligned} \gamma_5\,u_R(p) = u_R(p) \qquad &\Rightarrow \qquad \hat{\mathbf{p}}.\boldsymbol{\tau}\,\chi_R = \chi_R \\ \gamma_5\,u_L(p) = -u_L(p) \qquad &\Rightarrow \qquad \hat{\mathbf{p}}.\boldsymbol{\tau}\,\chi_L = -\chi_L, \end{aligned} \tag{3.26}$$

for right-handed and left-handed spinors respectively. Solving for $\hat{\mathbf{p}}.\boldsymbol{\tau}\,\chi = \pm\chi$, we get the 2-component spinors after proper normalisation:

$$\chi_R = \begin{pmatrix} \cos\frac{\theta}{2}\,e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}\,e^{i\frac{\phi}{2}} \end{pmatrix} \qquad\qquad \chi_L = \begin{pmatrix} -\sin\frac{\theta}{2}\,e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2}\,e^{i\frac{\phi}{2}} \end{pmatrix}, \tag{3.27}$$

and thus,

$$u_R(p) = \sqrt{\omega}\begin{pmatrix} \chi_R \\ \chi_R \end{pmatrix} \qquad\qquad u_L(p) = \sqrt{\omega}\begin{pmatrix} \chi_L \\ -\chi_L \end{pmatrix}, \tag{3.28}$$

One follows the same procedure for negative energy spinors, but since their momentum is $-p$ they satisfy

$$\gamma_5\,v_R(p) = -v_R(p), \qquad\qquad \gamma_5\,v_L(p) = v_L(p) \tag{3.29}$$

and, compared to the $u(p)$ spinors, the role of χ_R and χ_L is interchanged so that:

$$v_R(p) = \sqrt{\omega}\begin{pmatrix} -\chi_L \\ \chi_L \end{pmatrix} \qquad\qquad v_L(p) = -\sqrt{\omega}\begin{pmatrix} \chi_R \\ \chi_R \end{pmatrix}, \tag{3.30}$$

The relations $\chi_R^\dagger\chi_R = \chi_L^\dagger\chi_L = 1, \chi_R^\dagger\chi_L = \chi_L^\dagger\chi_R = 0$ ensure that eqs. (3.11) are satisfied.

• **Massive spinors : helicity and chirality**

In general, if in the rest-frame of the fermion the polarisation direction is given by the vector $s = (0, \mathbf{s})$ with $s^2 = -1, s.p = 0$, the spin projection operators along or opposite $\mathbf{s}$ are given, in a covariant form, by

$$\boxed{\Sigma^\pm(s) = \frac{1 \pm \gamma_5 \not{s}}{2}.} \tag{3.31}$$

Specifying to the helicity, the spin projection along or opposite the fermion momentum, one defines

$$\boxed{s = (\frac{p}{m}, \frac{\omega}{m}\hat{\mathbf{p}}), \qquad \text{with} \qquad p = |\mathbf{p}| \qquad \text{and} \qquad \hat{\mathbf{p}} = \frac{\mathbf{p}}{p},} \tag{3.32}$$

(which satisfies the conditions $s^2 = -1, s.p = 0$) and $\Sigma^\pm(s)$ takes the form:

$$\boxed{\Sigma^\pm(s) = \frac{1}{2m}\begin{pmatrix} m \pm \omega\hat{\mathbf{p}}.\boldsymbol{\tau} & \mp p \\ \pm p & m \mp \omega\hat{\mathbf{p}}.\boldsymbol{\tau} \end{pmatrix}.} \tag{3.33}$$

The form of the projectors $\Sigma^\pm(s)$ is different from the helicity projection operators defined in eq. (3.17) but when acting on positive energy spinors $u(p)$, one shows that:

$$\Sigma^\pm(s)\, u_\alpha(p) = \mathcal{S}^\pm(\hat{\mathbf{p}})\, u_\alpha(p), \qquad \alpha = 1, 2 \tag{3.34}$$

Thus, for positive energy spinors, Σ^+ projects out right-handed states and Σ^- the left-handed ones. Similarly, when acting on negative energy spinors $v(p)$, one finds that,

$$\Sigma^\pm(s)\, v_\alpha(p) = \mathcal{S}^\pm(-\hat{\mathbf{p}})\, v_\alpha(p), \qquad \alpha = 1, 2 \tag{3.35}$$

related to the fact that negative energy spinors carry momentum $-p$. Thus, again, Σ^+ projects out the right-handed helicity state and Σ^- the left-handed ones.
For massive spinors at very high energy if one uses $(1 \pm \gamma_5)/2$ as helicity projection operators rather than $\Sigma^\pm(s)$, with s as defined in eq. (3.32), the error made is of $\mathcal{O}(m/\omega)$[7].

[7] A negative chirality massive fermion at very high energy will be mainly left-handed with a small admixture, of $\mathcal{O}(m/\omega)$, of the right-handed component, and vice-versa.

In summary, it is easy to see that the fermion wave-functions:

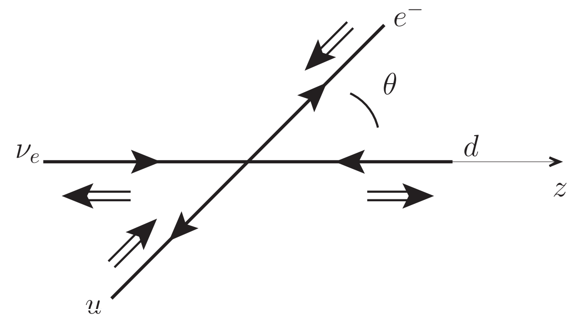
$$\begin{aligned}
\psi_R^f(p,x) &= \Sigma^+(s)\,\frac{\not p+m}{2m}\,\psi(p,x) && \text{destroys a right-handed fermion}\\
\psi_L^f(p,x) &= \Sigma^-(s)\,\frac{\not p+m}{2m}\,\psi(p,x) && \text{destroys a left-handed fermion}\\
\psi_R^{\bar f}(p,x) &= \Sigma^+(s)\,\frac{-\not p+m}{2m}\,\psi(p,x) && \text{creates a right-handed antifermion}\\
\psi_L^{\bar f}(p,x) &= \Sigma^-(s)\,\frac{-\not p+m}{2m}\,\psi(p,x) && \text{creates a left-handed antifermion.}
\end{aligned} \tag{3.36}$$

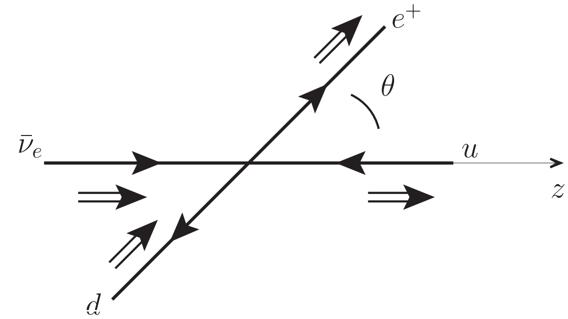
This summary will prove useful when discussing $\mathcal{C}$ and $\mathcal{CP}$ violation later.

Application

The helicity arguments above and conservation of angular momentum are useful to understand/predict the angular dependence of a process governed by the $\gamma_\mu(1-\gamma_5)$ interaction which carries total angular momentum 1 ($L=0, S=1$). For example, coming back to the processes $\nu_e\ d \to e^-\ u$ and $\overline{\nu}_e\ u \to e^+\ d$, eqs. (2.8) and (2.12), the leptonic transition is given by $\overline{\psi}_e\gamma_\mu(1-\gamma_5)\psi_{\nu_e} = 2\,\overline{\psi}_{e_L}\gamma_\mu\psi_{\nu_{eL}}$ or its hermitian conjugate $\overline{\psi}_{\nu_e}\gamma_\mu(1-\gamma_5)\psi_e = 2\,\overline{\psi}_{\nu_{eL}}\gamma_\mu\psi_{e_L}$. From eq. (3.23), we see that these transitions involve only left-handed leptons or right-handed antileptons. Likewise, from the $\overline{\psi}_d\gamma_\mu(1-\gamma_5)\psi_u$ or $\overline{\psi}_u\gamma_\mu(1-\gamma_5)\psi_d$ interactions, only left-handed quarks or right-handed antiquarks are allowed. In the scattering $\nu_e\ d \to e^-\ u$ only left-handed leptons and quarks are involved. If θ denotes the angle between the incoming and outgoing leptons in the $\nu\ d$ center of mass frame, the spin projection of the system along the axis of motion of the particles is 0 because each particle has a negative helicity and they move in opposite directions (see the figure). Therefore we expect no angular dependence for the cross section, in agreement with eq. (2.11). On the contrary, for the scattering $\overline{\nu}_e\ u \to e^+\ d$ the antileptons being right-handed and the quarks left-handed the spin projection of the antilepton-quark system along the direction of motion of the antilepton is always 1 : for a forward produced e^+ the angular momentum projection along the z axis is 1 for both initial and final states and thus is conserved while for a backward produced e^+ ($\theta=\pi$) the spin projection of the final system along the z axis is -1, and angular momentum is not conserved, consequently the matrix element vanishes. From Clebsh-Gordan tables[8] the associated angular distribution is proportional to $d^1_{11}(\theta) \simeq 1+\cos\theta$, in agreement with eq. (2.13).

[8]See, *Clebsh-Gordan coefficients, spherical harmonics and d-functions* in Particle Data group, C. Patrignani *et. al.*, Chin. Phys. **C40** (2016) 100001 (`http://pdg.lbl.gov`).





Similar arguments can be applied to $\nu/\overline{\nu}$ scattering on quarks or antiquarks and, then, one can easily derive eqs. (2.11), (2.13).

We note the useful relations :

$$\begin{aligned} \overline{\psi}_L \gamma_\mu \psi_L &= \frac{1}{2}\, \overline{\psi} \gamma_\mu (1-\gamma_5) \psi \\ \overline{\psi} \gamma_\mu \psi &= \overline{\psi}_L \gamma_\mu \psi_L + \overline{\psi}_R \gamma_\mu \psi_R, \qquad \overline{\psi}_L \gamma_\mu \psi_R = \overline{\psi}_R \gamma_\mu \psi_L = 0 \\ \overline{\psi} \psi &= \overline{\psi}_R \psi_L + \overline{\psi}_L \psi_R, \qquad \overline{\psi}_R \psi_R = \overline{\psi}_L \psi_L = 0. \end{aligned} \tag{3.37}$$