4 The global $\mathrm{SU(2)_L \otimes U(1)_Y}$ gauge invariance : conserved currents

Before entering the details of the model it is useful to recall the relation between rigid (global) gauge transformations and conserved currents since, as we shall see, the construction of the Weinberg-Salam model is made more transparent when using this notion. The choice of $\boldsymbol{SU(2)_L}$ is motivated by the structure of currents building up the Fermi interaction. For massless particles these currents are conserved, hence from Noether theorem, they are the consequences of a global $SU(2)$ invariance. The assumed $\boldsymbol{U(1)_Y}$ global invariance is the minimal group necessary to construct the electromagnetic current : indeed by an appropriate choice of the hypercharges Y, one constructs the electromagnetic current as the sum of the neutral $SU(2)$ one and the $U(1)$ current. Following the spectacular success of QED the $\boldsymbol{SU(2)_L \otimes U(1)_Y}$ invariance is made local to generate the interactions. It works !

4.1 Global gauge invariance and Noether theorem

One starts from the lagrangian density $\mathcal{L}(\psi(x), \partial_\mu \psi(x))$, which is a fonction of the field and its first derivatives, and from the action defined by

$$S = \int d^4x \mathcal{L}(\psi(x), \partial_\mu \psi(x)).$$

The action has no dimension. The Maupertuis principle (least action principle) states that, ”in Nature”, the action is stationary under a variation of the field and this leads to the Euler-Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta \psi(x)} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi(x)} = 0 \tag{4.1}$$

Now, assume that the lagrangian density is invariant under the rigid transformation

$$\begin{array}{rcl} \psi(x) & \to & e^{i\alpha}\psi(x) \\ \overline{\psi}(x) & \to & e^{-i\alpha}\overline{\psi}(x), \end{array} \tag{4.2}$$

where α is a real arbitrary constant, independent of the space-time coordinate x. Considering rather an infinitesimal transformation

$$\begin{array}{rcl} \delta\psi(x) & = & i\alpha\psi(x) \\ \delta\overline{\psi}(x) & = & -i\alpha\overline{\psi}(x), \end{array} \tag{4.3}$$

the variation of the lagrangian is[9] (note the relative position of the derivative term such as $\delta\mathcal{L}/\delta\psi$ and the $\delta\psi$)

$$\delta\mathcal{L}(\psi, \partial_\mu\psi) = \frac{\delta\mathcal{L}}{\delta\psi}\delta\psi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\delta\partial_\mu\psi + \delta\overline{\psi}\frac{\delta\mathcal{L}}{\delta\overline{\psi}} + (\delta\partial_\mu\overline{\psi})\frac{\delta\mathcal{L}}{\delta\partial_\mu\overline{\psi}}. \tag{4.4}$$

[9] To lighten the notation and when no ambiguity arises one simply writes in the following ψ for $\psi(x)$ and $\delta\psi$ for $\delta\psi(x)$.

But, $\delta\partial_\mu\psi = \partial_\mu\delta\psi = i\alpha\partial_\mu\psi$, $\delta\partial_\mu\overline{\psi} = \partial_\mu\delta\overline{\psi} = -i\alpha\partial_\mu\overline{\psi}$, and using the Euler-Lagrange equations to eliminate $\delta\mathcal{L}/\delta\psi$ and $\delta\mathcal{L}/\delta\overline{\psi}$ one finds

$$\delta\mathcal{L}(\psi,\partial_\mu\psi) = i\alpha\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\psi - \overline{\psi}\frac{\delta\mathcal{L}}{\delta\partial_\mu\overline{\psi}}\right). \tag{4.5}$$

Since $\delta\mathcal{L}(\psi,\partial_\mu\psi) = 0$ under the variation of the fields, eqs. (4.3), the current defined by

$$\boxed{J^\mu(x) = \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\psi - \overline{\psi}\frac{\delta\mathcal{L}}{\delta\partial_\mu\overline{\psi}}} \tag{4.6}$$

is conserved, *i.e.* $\boxed{\partial_\mu J^\mu(x) = 0}$. For a fermion field with the langrangien density

$$\mathcal{L}(\psi,\partial_\mu\psi) = \overline{\psi}(i\,\partial\!\!\!/ - m)\psi$$

the conserved current is simply

$$J^\mu(x) = \overline{\psi}\gamma^\mu\psi. \tag{4.7}$$

One defines the charge by the space integration of the 0th component of the current, and specifying $x = (t,\mathbf{x}), d^4x = dtd^3x$, one has

$$\boxed{Q(t) = \int d^3x J^0(t,\mathbf{x}) = \int d^3x\psi^\dagger(t,\mathbf{x})\psi(t,\mathbf{x}).} \tag{4.8}$$

Using current conservation, $\partial_\mu J^\mu(x) \equiv \partial_t J^0(t,\mathbf{x}) + \boldsymbol{\nabla}\cdot\mathbf{J}(t,\mathbf{x}) = 0$, it is easy to prove that the charge is time independent since

$$\frac{dQ(t)}{dt} = -\int_\Omega d^3x\boldsymbol{\nabla}\cdot\mathbf{J}(t,\mathbf{x}) = -\int_{\partial\Omega}\boldsymbol{ds}\cdot\mathbf{J}(t,\mathbf{x}) = 0, \tag{4.9}$$

where the last equality is realised when we assume the fields are suppressed at infinity.

Thus the Noether theorem states that to an invariance under a set of continuous transformations corresponds a conserved current. The results eqs. (4.6), (4.7) are easily extended to the case of non abelian symmetries such as $SU(2),\cdots,SU(N)$ or to a lagrangian density involving several fields ψ_i. Then if

$$\mathcal{L}(\psi,\partial_\mu\psi) = \sum_i \overline{\psi}_i(i\,\partial\!\!\!/ - m_i)\psi_i \tag{4.10}$$

is invariant under the set of transformations

$$\begin{aligned} \delta\psi_i &= i\,\alpha\,y_i\,\psi_i, \quad \forall i \\ \delta\overline{\psi}_i &= -i\,\alpha\,y_i\,\overline{\psi}_i, \quad \forall i, \end{aligned} \tag{4.11}$$

with α a common real parameter and y_i the charge of field ψ_i, the conserved current is

$$\boxed{J^\mu(x) = \sum_i \, y_i \, \overline{\psi}_i \gamma^\mu \psi_i,} \tag{4.12}$$

a result to be used later. The charge of fermion ψ_i is then defined as y_i.

4.2 The lagrangian density

As discussed above, the weak interactions induce a transition between left-handed fermions of different charges. It is then natural to group them into doublets

$$\underbrace{\begin{pmatrix} \nu_{e_L} \\ e^-_{_L} \end{pmatrix}, \begin{pmatrix} \nu_{\mu_L} \\ \mu^-_{_L} \end{pmatrix}, \begin{pmatrix} \nu_{\tau_L} \\ \tau^-_{_L} \end{pmatrix}}_{\text{leptons}} ; \underbrace{\begin{pmatrix} u_{_L} \\ d_{_L} \end{pmatrix}, \begin{pmatrix} c_{_L} \\ s_{_L} \end{pmatrix}, \begin{pmatrix} t_{_L} \\ b_{_L} \end{pmatrix}}_{\text{quarks}} \tag{4.13}$$

We introduce the left handed doublets:

$$\Psi_{e_L} = \begin{pmatrix} \nu_{e_L} \\ e_{_L} \end{pmatrix}, \cdots \qquad \Psi_{q_L} = \begin{pmatrix} u_{_L} \\ d_{_L} \end{pmatrix}, \cdots, \tag{4.14}$$

and the right-handed singlets $\psi_{e_R} = e_{_R}, \cdots, \psi_{q_R} = q_{_R}, \cdots$. We assume all fermions are massless. The free massless fermion lagrangian is then written,

$$-i\mathcal{L}_F = \bar{e}\not\partial \, e + \bar{\nu}_{e_L}\not\partial \, \nu_{e_L} + \ \bar{u}\not\partial \, u + \bar{d}\not\partial \, d \tag{4.15}$$

where we have kept the first family of fermions ν_e, e, u, d only and where we have ignored the right-handed neutrino ν_{e_R} not observed experimentally. Using the second of eq. (3.37) and regrouping the members of a doublet one finds

$$-i\mathcal{L}_F = \overline{\Psi}_{e_L}\not\partial \, \Psi_{e_L} + \overline{\Psi}_{q_L} \ \not\partial \, \Psi_{q_L} + \overline{e_{_R}} \ \not\partial \, e_{_R} + \overline{u_{_R}}\not\partial \, u_{_R} + \overline{d_{_R}}\not\partial \, d_{_R} \tag{4.16}$$

For massless fermions the charged current introduced by Fermi, eq. (2.2), is conserved so it is tempting to introduce a global symmetry associated to this current.

4.3 The global $SU(2)_L$ gauge invariance

It is obvious that the lagrangian above is invariant under a global SU(2) phase change of the left handed fermion fields, *i.e.* under the transformation,

$$\Psi_{_L} \to \mathrm{e}^{i\,\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2}\Psi_{_L} \quad , \quad \overline{\Psi}_{_L} \to \overline{\Psi}_{_L}\mathrm{e}^{-i\,\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/\mathbf{2}} \tag{4.17}$$

where the 2×2 Pauli matrices $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ satisfy the algebra

$$\left[\frac{\tau_i}{2}, \frac{\tau_j}{2}\right] = i\ \epsilon_{ijk}\ \frac{\tau_k}{2}, \tag{4.18}$$

and have the following properties

$$\tau = \tau^\dagger, \qquad \mathrm{Tr}(\tau_i \tau_j) = 2\delta_{ij} \tag{4.19}$$

The parameter $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is a set of 3 arbitrary constants. As discussed above to a global symmetry is associated a conserved current. The $SU(2)$ group has three generators and there are three conserved currents. Following the reasoning leading to eq. (4.6) they are identified to

$$\boxed{J_i^\mu(x) = \overline{\Psi}_{e_L} \gamma^\mu \frac{\tau_i}{2} \Psi_{e_L} + \overline{\Psi}_{q_L} \gamma^\mu \frac{\tau_i}{2} \Psi_{q_L}.} \tag{4.20}$$

They are called the "weak isospin currents". The first two, $J_1^\mu(x)$, $J_2^\mu(x)$, are related to the currents introduced by E. Fermi to describe the weak interaction: for example, using the first of the eq. (3.37) identities, $J_1^\mu(x)$ is written

$$\begin{aligned} J_1^\mu(x) &= \frac{1}{2}(\overline{e}_L \gamma^\mu \nu_{e_L} + \overline{d}_L \gamma^\mu u_L + \text{ h.c.}), \\ &= \frac{1}{4}(\overline{e} \gamma^\mu (1-\gamma_5) \nu_e + \overline{d} \gamma^\mu (1-\gamma_5) u + \text{ h.c.}), \end{aligned}$$

which together with $J_2^\mu(x)$ allows to reconstruct eq. (2.2). The third one is new, it is a neutral current,

$$J_3^\mu(x) = \frac{1}{2}[\overline{\nu}_{e_L} \gamma^\mu \nu_{e_L} - \overline{e}_L \gamma^\mu e_L + \overline{u}_L \gamma^\mu u_L - \overline{d}_L \gamma^\mu d_L]. \tag{4.21}$$

The corresponding weak isopin charge is given by,

$$I_3 = \int d^3x J_3^0(x) = \frac{1}{2} \int d^3x (\nu_{e_L}^\dagger \nu_{e_L} - e_L^\dagger e_L + u_L^\dagger u_L - d_L^\dagger d_L), \tag{4.22}$$

which allows to assign a charge $I_3 = +1/2$ to the neutrino and the u quark and $I_3 = -1/2$ to the electron and the d quark. Obviously $J_3^\mu(x)$ cannot be the current coupling to the photon field otherwise the neutrino would interact with the photon ! $J_3^\mu(x)$ is a neutral current since it does not change the charge of the fermion.

4.4 The global $U(1)_Y$ gauge invariance

The lagrangian $\mathcal{L}_F$ of eq. (4.15) is invariant under a $U(1)$ global transformation acting on all fields, left and right. It is called called the $U(1)_Y$ group, where Y refers to the hypercharge. A transformation is defined by :

$$\begin{aligned}
\Psi_{e_L} &\to \mathrm{e}^{i\,\beta y^e_L/2}\Psi_{e_L}, &\qquad \Psi_{q_L} &\to \mathrm{e}^{i\,\beta y^q_L/2}\Psi_{q_L} \\
e_R &\to \mathrm{e}^{i\,\beta y^e_R/2} e_R, && \\
u_R &\to \mathrm{e}^{i\,\beta y^u_R/2} u_R, &\qquad d_R &\to \mathrm{e}^{i\,\beta y^d_R/2} d_R,
\end{aligned} \tag{4.23}$$

where the $y^e_L, y^q_L, y^e_R, y^u_R, y^d_R$ are the hypercharges of the corresponding fields. The associated conserved current writes (see eq. (4.12))

$$\boxed{J^\mu_Y(x) = y^e_L \overline{\Psi}_{e_L}\gamma^\mu \Psi_{e_L} + y^q_L \overline{\Psi}_{q_L}\gamma^\mu \Psi_{q_L} + y^e_R \overline{e}_R \gamma^\mu e_R + y^u_R \overline{u}_R \gamma^\mu u_R + y^d_R \overline{d}_R \gamma^\mu d_R.} \tag{4.24}$$

Since the sum of conserved currents is also a conserved current we can construct the electromagnetic current,

$$\boxed{J^\mu_{\mathrm{emg}}(x) = e_e \overline{e}\gamma^\mu e + e_u \overline{u}\gamma^\mu u + e_d \overline{d}\gamma^\mu d,} \tag{4.25}$$

as the sum of the weak isospin and hypercharge currents[10] :

$$\boxed{J^\mu_{\mathrm{emg}}(x) = J^\mu_3(x) + \frac{J^\mu_Y(x)}{2},} \tag{4.26}$$

with the hypercharges of the fields chosen so as to construct their correct electric charges (which are normalised here to the charge of the proton). For the lepton sector, for exemple, one finds -1 for the left-handed doublet and -2 for the right-handed electron partner to get a charge of -1 for both left-handed and right-handed component of the electron and 0 for the neutrino. The results are summarised in the following table :

	I	I_3	Y	Q
ν_e	1/2	1/2	-1	0
e_L	1/2	$-1/2$	-1	-1
e_R	0	0	-2	-1
u_L	1/2	1/2	1/3	2/3
d_L	1/2	$-1/2$	1/3	$-1/3$
u_R	0	0	4/3	2/3
d_R	0	0	$-2/3$	$-1/3$

(4.27)

which shows that the relation between the charge, hypercharge and weak isospin satisfies, by construction, the famous Gell-Mann/Nishijima relation :

$$\boxed{Q = I_3 + \frac{Y}{2}} \tag{4.28}$$

[10] the facteur 1/2 associated to the hypercharge current is historically conventional.

• **Application**

In general, for a $SU(2)$ doublet $\Phi^T = (\phi_1, \phi_2)$ of fields of hypergharge y_{Φ} and electric charges (e_1, e_2), the Gell-Mann/Nishijima relation yields

$$e_1 - e_2 = 1 \quad \text{and} \quad y_{\Phi} = e_1 + e_2, \tag{4.29}$$

Thus the charges of the members of a doublet always differ by one unit of charge while the hypercharge is the sum of the electric charges. For a field singlet under $SU(2)$ the relation between hypercharge and electric charge is simply

$$e_{\phi} = \frac{y_{\phi}}{2}. \tag{4.30}$$