5 The local $\mathbf{SU(2)_L \otimes U(1)_Y}$ gauge invariance : interactions

The local SU(2) transformation, acting on the left-handed doublets only, is defined by

$$\Psi'_L \to U(x)\Psi_L = \mathrm{e}^{i\,g\,\boldsymbol{\alpha}(x)\cdot\boldsymbol{\tau}/2}\Psi_L \quad , \quad \overline{\Psi'}_L \to \overline{\Psi}_L U^\dagger(x) = \overline{\Psi}_L \mathrm{e}^{-i\,g\,\boldsymbol{\alpha}(x)\cdot\boldsymbol{\tau}/2}, \tag{5.1}$$

with $UU^\dagger = 1$, or, for an infinitesimal transformation,

$$\delta\Psi_L = i\ g\ \boldsymbol{\alpha}(x)\cdot\frac{\boldsymbol{\tau}}{2}\Psi_L \quad , \quad \delta\overline{\Psi}_L = -i\ g\ \overline{\psi}_L\boldsymbol{\alpha}(x)\cdot\frac{\boldsymbol{\tau}}{2}, \tag{5.2}$$

where the 3 components of the real parameter $\boldsymbol{\alpha}(x)$ are functions of the space-time coordinates. We have introduced a coupling g associated to this transformation. Under the local transformation the lagrangian density (4.15) is no longer invariant because of the derivative term in $\partial^\mu\boldsymbol{\alpha}(x)$

$$\delta\mathcal{L}_F = \overline{\Psi}_{e_L}\{-g(\partial^\mu\boldsymbol{\alpha}(x))\cdot\frac{\boldsymbol{\tau}}{2}\}\gamma_\mu\Psi_{e_L} + \overline{\Psi}_{q_L}\{-g(\partial^\mu\boldsymbol{\alpha}(x))\cdot\frac{\boldsymbol{\tau}}{2}\}\gamma_\mu\Psi_{q_L} \tag{5.3}$$

To recover the invariance of $\mathcal{L}_F$ under this transformation one introduces a multiplet (a triplet) of gauge vector fields $\mathbf{W}^\mu(x) = (W_1^\mu(x), W_2^\mu(x), W_3^\mu(x))$ and defines the covariant derivative operating only on the left-handed fields :

$$D_L^\mu = \partial^\mu - ig\mathcal{W}^\mu(x), \qquad \text{with} \qquad \mathcal{W}^\mu(x) = \frac{\boldsymbol{\tau}}{2}\cdot\mathbf{W}^\mu(x). \tag{5.4}$$

The transformation properties of $\mathbf{W}^\mu(x)$ are chosen such that the lagrangian density

$$\mathcal{L}_F = \overline{\Psi}_{e_L}\not{D}_L\ \Psi_{e_L} + \overline{\Psi}_{q_L}\ \not{D}_L\ \Psi_{q_L} + \overline{\psi}_{e_R}\ \not{\partial}\ \psi_{e_R} + \overline{\psi}_{u_R}\not{\partial}\ \psi_{u_R} + \overline{\psi}_{d_R}\not{\partial}\ \psi_{d_R} \tag{5.5}$$

is invariant under an $SU(2)$ transformation. Since the right-handed fields are not affected by the transformation it is enough to impose that $D_L^\mu\Psi(x)$ transforms as $\Psi(x)$ to achieve the invariance of the lagrangian:

$$(D_L^\mu\Psi(x))' = U(x)(D_L^\mu\Psi(x)). \tag{5.6}$$

Therefore,

$$(D_L^\mu\Psi(x))' = (D_L^\mu)'U(x)\Psi(x)) = U(x)D_L^\mu\Psi(x), \tag{5.7}$$

implies

$$(D_L^\mu)' = U(x)D_L^\mu U^{-1}(x), \tag{5.8}$$

since it should hold for all $\Psi(x)$. Consequently, using $\partial^\mu U^{-1}(x) = (\partial^\mu U^{-1}(x)) + U^{-1}(x)\partial^\mu$, one finds

$$(D_L^\mu)' = \partial^\mu + U(x)(\partial^\mu U^{-1}(x)) - igU(x)\mathcal{W}^\mu(x)U^{-1}(x), \tag{5.9}$$

which can be written as $(D_L^\mu)' = \partial^\mu - ig\mathcal{W}'^\mu(x)$ with

$$\mathcal{W}'^\mu(x) = \frac{i}{g}U(x)(\partial^\mu U^{-1}(x)) + U(x)\mathcal{W}^\mu(x)U^{-1}(x) \tag{5.10}$$

Restricting to the infinitesimal transformations eq. (5.2), one obtains

$$\boxed{\mathcal{W}'^\mu(x) - \mathcal{W}^\mu(x) = \delta\mathcal{W}^\mu(x) = \partial^\mu \boldsymbol{\alpha}(x)\cdot\frac{\boldsymbol{\tau}}{2} + ig\,[\boldsymbol{\alpha}(x)\cdot\frac{\boldsymbol{\tau}}{2}, \mathcal{W}^\mu(x)],} \tag{5.11}$$

which, in terms of $SU(2)$ components, is equivalent to

$$\boxed{\delta\, W_i^\mu(x) = \partial^\mu\alpha_i(x) - g\,\epsilon_{ijk}\,\alpha_j(x)\,W_k^\mu(x).} \tag{5.12}$$

spinor we have $D_L^{'\mu}U = UD_L^\mu$, hence eq. (5.8).

To construct the kinetic term of the gauge bosons $W_i^\mu(x)$ we first consider, as in QED, the tensor

$$\mathcal{F}^{\mu\nu}(x) = [D_L^\mu(x), D_L^\nu(x)] \tag{5.13}$$

Using Leibnitz rule $\partial_\mu W_i^\nu(x) = (\partial_\mu W_i^\nu(x)) + W_i^\nu(x)\partial_\mu$ it is easy to show that the tensor is given by

$$\mathcal{F}^{\mu\nu}(x) = \partial^\mu\mathcal{W}^\nu(x) - \partial^\nu\mathcal{W}^\mu(x) - ig[\mathcal{W}^\mu(x), \mathcal{W}^\nu(x)] \tag{5.14}$$

or in components

$$\boxed{F_i^{\mu\nu}(x) = \partial^\mu W_i^\nu(x) - \partial^\nu W_i^\mu(x)\ + g\,\epsilon_{ijk}\,W_j^\mu(x)W_k^\nu(x).} \tag{5.15}$$

The transformation property of $\mathcal{F}^{\mu\nu}(x)$ is obviously the same as that of D_L^μ, eq. (5.8), and we have then $\mathcal{F}'^{\mu\nu}(x) = U\mathcal{F}^{\mu\nu}(x)U^{-1}$ so that

$$\mathrm{Tr}\mathcal{F}^{\mu\nu}(x)\mathcal{F}_{\mu\nu}(x) = \frac{1}{2}F_i^{\mu\nu}(x)F_{i\mu\nu}(x) \tag{5.16}$$

is a Lorentz scalar invariant under a gauge transformation by the property of cyclicity of the trace. Furthermore it has the right dimension to be the kinetic term of the W_i^μ bosons. The lagrangian of left-handed fields becomes then :

$$\mathcal{L}_{F_L} = -\frac{1}{4}\,F_i^{\mu\nu}(x)F_{i\mu\nu}(x) + \overline{\Psi}_{e_L} iD_L^\mu\gamma_\mu\Psi_{e_L} + \overline{\Psi}_{q_L} iD_L^\mu\gamma_\mu\Psi_{q_L}. \tag{5.17}$$

where each of the three terms is invariant under a local $SU(2)$ transformation. We note at this point the perfect analogy between the construction of the "weak" lagrangian with that of QCD: the differences are in the choice of group which requires here only three vector bosons while for $SU(3)$ symmetry eight bosons had to be introduced. Also, the $SU(2)$ group acts only on the left handed components of the fields and consequently the $W_i^\mu(x)$ gauge bosons do not couple to the right handed fermion components.

We now make the $U(1)_Y$ gauge transformation local. It is defined by

$$\begin{aligned}
\delta\Psi_{e_L} &= ig'\frac{y_L^e}{2}\beta(x)\ \Psi_{e_L}, \qquad & \delta\Psi_{q_L} &= ig'\frac{y_L^q}{2}\beta(x)\ \Psi_{q_L} \\
\delta e_R &= ig'\frac{y_R^e}{2}\beta(x)\ e_R, & & \\
\delta u_R &= ig'\frac{y_R^u}{2}\beta(x)\ u_R, & \delta d_R &= ig'\frac{y_R^d}{2}\beta(x)\ d_R,
\end{aligned} \tag{5.18}$$

with g' the coupling associated to the $U(1)$ transformation. To keep the invariance of the lagrangien requires the introduction of another vector boson $B_\mu(x)$ to which are associated covariant derivatives generating couplings of $B_\mu(x)$ to fermions. Because the fermions carry different hypercharges we introduce covariant derivatives appropriate for each right-handed field : acting on field ψ_R ($\psi = e, u, d$) it is[11]

$$\boxed{D^\mu_{\psi_R} = \partial^\mu - i\ g'\ \frac{y^{\psi_R}}{2}\ B^\mu,} \tag{5.19}$$

while for the left handed fields the covariant derivative eq. (5.4) acquires a new piece and becomes :

$$\boxed{D^\mu_{\psi_L} = \partial^\mu - i\ g\ \frac{\boldsymbol{\tau}}{2}\cdot\mathbf{W}^\mu - i\ g'\ \frac{y^{\psi_L}}{2}\ B^\mu.} \tag{5.20}$$

The stress-energy tensor of the new vector field is simply :

$$\boxed{\mathcal{K}^{\mu\nu}(x) = \partial^\mu B^\nu(x) - \partial^\nu B^\mu(x)} \qquad \text{(abelian field).} \tag{5.21}$$

In summary, the initial free lagrangian eq. (4.15) becomes, after imposing a $SU(2)$ local symmetry on the left-handed fields and an appropriate $U(1)$ invariance on both the left-handed fields and a right-handed ones,

$$\boxed{\begin{aligned}
\mathcal{L} = \mathcal{L}_G\ +\ \mathcal{L}_F = &-\frac{1}{4}\ F_{i\mu\nu}(x)\ F_i^{\mu\nu}(x) - \frac{1}{4}\ \mathcal{K}_{\mu\nu}(x)\ \mathcal{K}^{\mu\nu}(x) \\
&+\ \overline{\Psi}_{e_L}\ i\ \not{D}_{e_L}\ \Psi_{e_L} + \overline{\Psi}_{q_L}\ i\ \not{D}_{q_L}\ \Psi_{q_L} + \\
&+\ \overline{e}_R\ i\ \not{D}_{e_R}\ e_R + \overline{u}_R\ i\ \not{D}_{u_R}\ u_R + \overline{d}_R\ i\ \not{D}_{d_R}\ d_R
\end{aligned}} \tag{5.22}$$

where only the (e, ν_e) and (u, d) quark family has been specified. It is important to point out that the $SU(2)_L \otimes U(1)_Y$ invariance imposes that all fermions are massless. Indeed a fermion mass term

[11]The left and right covariant derivatives generically defined as D^μ_L, D^μ_R are now denoted $D^\mu_{\psi_L}$, $D^\mu_{\psi_R}$ since they depend on the quantum numbers of the fermion fields ψ_L, ψ_R.

in the lagrangian would have the form

$$\mathcal{L}_{mass} = m\,\overline{\psi}\psi = m(\overline{\Psi}_L \psi_R \ + \ \overline{\psi}_R \Psi_L). \tag{5.23}$$

But since Ψ_L is a doublet and $\overline{\psi}_R$ a singlet under $SU(2)$, the mass term cannot be invariant under a gauge transformation!

It is useful to separate the lagrangian density eq. (5.22) into a free part

$$\begin{aligned}\mathcal{L}_{0F} + \mathcal{L}_{0G} \ =\ & \overline{\Psi}_{e_L}\ i\ \partial\!\!\!/\ \Psi_{e_L} + \overline{\Psi}_{q_L}\ i\ \partial\!\!\!/\ \Psi_{q_L} + \ \overline{e}_R\ i\ \partial\!\!\!/\ e_R + \overline{u}_R\ i\ \partial\!\!\!/\ u_R + \overline{d}_R\ i\ \partial\!\!\!/\ d_R \qquad (5.24)\\ & -\frac{1}{4}[(\partial_\mu \mathbf{W}_\nu(x) - \partial_\nu \mathbf{W}_\mu(x)) \cdot (\partial^\mu \mathbf{W}^\nu(x) - \partial^\nu \mathbf{W}^\mu(x)) + (\partial_\mu B_\nu(x) - \partial_\nu B_\mu(x))\ (\partial^\mu B^\nu(x) - \partial^\nu B^\mu(x))],\end{aligned}$$

and an interacting part containing all terms depending on the couplings g and g'. It contains two classes of terms : one describing the fermion-gauge bosons interactions (which can be expressed very easily in terms of the currents introduced above) and the other the W boson self interactions

$$\begin{aligned}\mathcal{L}_{IF} + \mathcal{L}_{IG} \ =\ & g\ \mathbf{J}^\mu(x) \cdot \mathbf{W}_\mu(x) + g' \frac{J_Y^\mu(x)}{2} B_\mu(x) \qquad (5.25)\\ & -\frac{g}{2}\epsilon_{ijk}(\partial_\mu W_{i\nu}(x) - \partial_\nu W_{i\mu}(x)) W_j^\mu(x) W_k^\nu(x) - \frac{g^2}{4}\epsilon_{ijk} W_{j\mu}(x) W_{k\nu}(x)\ \epsilon_{ilm} W_l^\mu(x) W_m^\nu(x)\end{aligned}$$

with $\mathbf{J}^\mu$ the weak isospin current of eq. (4.20) and J_Y^μ the hypercharge current of eq. (4.24). One recognizes in the sum of these two terms the expression which lead to the construction of the electromagnetic current in eq. (4.25).

5.1 Fermion-boson interactions, construction of the photon and the Z boson

We turn first to the fermion-$\mathbf{W}^\mu$ interaction. It is read off from $\mathcal{L}_{IF}$ and is simply

$$g\ \mathbf{J}^\mu(x) \cdot \mathbf{W}_\mu(x) = \frac{g}{2}\left(\overline{\Psi}_{e_L} \gamma^\mu \tau_i \Psi_{e_L} + \overline{\Psi}_{q_L} \gamma^\mu \tau_i \Psi_{q_L}\right) W_{i\mu}. \tag{5.26}$$

Defining the charged vector fields

$$(W^\pm)^\mu(x) = \frac{(W_1^\mu(x) \mp i W_2^\mu(x))}{\sqrt{2}}, \qquad \text{with} \qquad (W^{+*})^\mu(x) = (W^-)^\mu(x) \tag{5.27}$$

their interaction with the fermions can be easily obtained from the charge changing part of the currents $(J_1^\mu(x), J_2^\mu(x))$ in eq. (5.26) and we find

$$\begin{aligned}\mathcal{L}_{IF}(\text{charged current}) \ &=\ \frac{g}{\sqrt{2}}(\overline{\nu}_{e_L}\,\gamma^\mu\, e_L\, W_\mu^+ + \overline{u}_L\,\gamma^\mu\, d_L\, W_\mu^+ + \text{ h.c.}) \qquad (5.28)\\ &=\ \frac{g}{2\sqrt{2}}(\overline{\nu}_e \gamma^\mu (1-\gamma_5)\, e\, W_\mu^+ + \overline{u}\,\gamma^\mu (1-\gamma_5)\, d\, W_\mu^+ + \text{ h.c.}), \qquad (5.29)\end{aligned}$$

which is now expressed in terms of the usual fermion fields $\nu_e, e, u,\ d$. One can thus read off the $W^{\pm}$ coupling to fermions : using standard techniques it is found to be $-i(g/2\sqrt{2})\gamma_\mu(1-\gamma_5)$, coupling with the same strength to all fermion species. (Note the relation $g/2\sqrt{2} = g_W$ of eq. (2.16)).

Turning now to the neutral vector bosons sector one has two pieces : one originates from the $SU(2)_L$ invariance, namely $gJ_3^\mu W_{3\mu}$ contained in eq. (5.26), and the other one from the $U(1)_Y$ invariance, $g' J_Y^\mu B_\mu$. From eq. (5.25) we can read off the neutral current interaction lagrangian which is

$$\mathcal{L}_{IF}(\text{neutral currents}) = gJ_3^\mu W_{3\mu} + g'\frac{1}{2}J_Y^\mu B_\mu \tag{5.30}$$

Note that the photon cannot be identified to the $W_{3\mu}$ field because of the γ_5 term in the coupling nor to the B_μ boson because of the different charge assignment for the left and right component of a fermion field. The photon will be constructed as a linear combination of both. Thus, introducing the fields A_μ and Z_μ such that

$$\begin{aligned} B^\mu &= \cos\theta\ A^\mu - \sin\theta\ Z^\mu \\ W_3^\mu &= \sin\theta\ A^\mu + \cos\theta\ Z^\mu, \end{aligned} \tag{5.31}$$

with θ an adjustable parameter, one finds

$$\mathcal{L}_{IF}(\text{neutral currents}) = (g\sin\theta J_3^\mu + g'\cos\theta\frac{1}{2}J_Y^\mu)A_\mu + (g\cos\theta J_3^\mu - g'\sin\theta\frac{1}{2}J_Y^\mu)Z_\mu. \tag{5.32}$$

To construct the field A_μ as the photon field we should adjust the parameters to be such that

$$g\sin\theta J_3^\mu + g'\cos\theta\frac{1}{2}J_Y^\mu = eJ_{\text{emg}}^\mu \tag{5.33}$$

where, by convention, e is taken as the charge of the proton. This can be achieved if we choose

$$\boxed{g\sin\theta = g'\cos\theta = e} \tag{5.34}$$

since, then, we recover eq. (4.26) which lead to eq. (4.25) for J_{emg}^μ. With this choice, we have $J_Y^\mu/2 = J_{\text{emg}}^\mu - J_3^\mu$ which is used to eliminate in the coefficient of Z_μ the hypercharge current so that the interaction lagrangien reads

$$\mathcal{L}_{IF}(\text{neutral currents}) = eJ_{\text{emg}}^\mu A_\mu + \frac{e}{\sin\theta\cos\theta}(J_3^\mu - \sin^2\theta J_{\text{emg}}^\mu)Z_\mu, \tag{5.35}$$

defining the couplings of the photon $A_\mu(x)$ and the neutral $Z_\mu(x)$ boson to the fermions. Concerning the Z_μ couplings we can be more explicit and derive them for a pair of fermions ψ_1, ψ_2 of charge

e_1, e_2 (normalised to the proton charge e) respectively, such that (ψ_{1L}, ψ_{2L}) forms a $SU(2)$ doublet $(I = 1/2)$ and ψ_{1R}, ψ_{2R} are singlets $(I = 0)$. Writing explicitely the currents J_3^μ and $J_{\rm emg}^\mu$, we have from eq. (5.35):

$$
\begin{aligned}
&\frac{e}{\sin\theta\cos\theta}\Bigg[(\overline{\psi}_{1_L}\ \overline{\psi}_{2_L})\begin{pmatrix} 1/2 - e_1\sin^2\theta & 0 \\ 0 & -1/2 - e_2\sin^2\theta \end{pmatrix} \not{Z} \begin{pmatrix} \psi_{1_L} \\ \psi_{2_L} \end{pmatrix} \\
&\qquad + (\overline{\psi}_{1_R}\ \overline{\psi}_{2_R})\begin{pmatrix} -e_1\sin^2\theta & 0 \\ 0 & -e_2\sin^2\theta \end{pmatrix} \not{Z} \begin{pmatrix} \psi_{1R} \\ \psi_{2R} \end{pmatrix}\Bigg] \\
&= \frac{e}{\sin\theta\cos\theta}\Bigg[(\overline{\psi}_1\ \overline{\psi}_2)\begin{pmatrix} 1/2 - e_1\sin^2\theta & 0 \\ 0 & -1/2 - e_2\sin^2\theta \end{pmatrix} \not{Z}\frac{(1-\gamma_5)}{2} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
&\qquad + (\overline{\psi}_1\ \overline{\psi}_2)\begin{pmatrix} -e_1\sin^2\theta & 0 \\ 0 & -e_2\sin^2\theta \end{pmatrix} \not{Z}\frac{(1+\gamma_5)}{2} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\Bigg] \\
&= \frac{e}{\sin\theta\cos\theta}\Bigg[(\overline{\psi}_1\ \overline{\psi}_2)\begin{pmatrix} 1/4 - e_1\sin^2\theta & 0 \\ 0 & -1/4 - e_2\sin^2\theta \end{pmatrix} \not{Z} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
&\qquad - (\overline{\psi}_1\ \overline{\psi}_2)\begin{pmatrix} 1/4 & 0 \\ 0 & -1/4 \end{pmatrix} \not{Z}\,\gamma_5 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\Bigg]
\end{aligned}
\tag{5.36}
$$

The full neutral current interaction lagrangian density eq. (5.35) can then be written for one generation of quarks and leptons

$$
\boxed{
\begin{aligned}
\mathcal{L}_{IF}(\text{neutral currents}) = -e\ \overline{e}\not{A}\ e + &\frac{e}{\sin\theta\cos\theta}\sum_{l=\nu,e}\bar{l}\not{Z}\ (a_l - b_l\gamma_5)\,l \\
+\,e\sum_{q=u,d} e_q\ \overline{q}\not{A}\ q\ + &\frac{e}{\sin\theta\cos\theta}\sum_{q=u,d}\overline{q}\not{Z}\ (a_q - b_q\gamma_5)q
\end{aligned}}
\tag{5.37}
$$

with

$$
\boxed{a_i = \frac{I_3}{2} - e_i\sin^2\theta, \qquad b_i = \frac{I_3}{2}.}
\tag{5.38}
$$

Contrary to the photon which has a purely vector coupling to the fermions, the neutral gauge boson Z_μ has both vector and axial-vector couplings. We recall that with the choice of $g = e/\cos\theta$ the charged W_μ couplings are

$$
\boxed{\mathcal{L}_{IF}(\text{charged current}) = \frac{e}{2\sqrt{2}\sin\theta}(\overline{\nu}_e\gamma^\mu(1-\gamma_5)\,e\,W_\mu^+ + \overline{u}\,\gamma^\mu(1-\gamma_5)\,d\,W_\mu^+ + \text{ h.c.}),}
\tag{5.39}
$$

These couplings are in agreement with those of the physical Z boson once the "weak mixing" or Weinberg angle θ (in fact introduced by Glashow!) is taken from experiment to be :

$$
\sin^2\theta \sim .2313\ .
\tag{5.40}
$$

We herafter denote the weak mixing angle by θ_W.

• **The covariant derivative in terms of the $A_\mu, Z_\mu, W^\pm_\mu$ fields**

It is useful, for later use, to have an explicit representation of the covariant derivatives eqs. (5.19) and (5.20) in terms of the $W^\pm_\mu, A_\mu$ and Z_μ gauge bosons. Although they can be read off the previous discussion based on defining the electromagnetic current we construct them directly. For instance, the covariant derivative eq. (5.20) acting on a $SU(2)$ doublet of fields with hypercharge y_ϕ, the components of which having electric charge (ee_1, ee_2), contains the piece

$$-ig\frac{\tau_3}{2}W_{3\mu} - ig'\frac{y_\phi}{2}B_\mu = -i\left[(g\sin\theta_W\frac{\tau_3}{2} + g'\cos\theta_W\frac{y_\phi}{2})A_\mu + (g\cos\theta_W\frac{\tau_3}{2} - g'\sin\theta_W\frac{y_\phi}{2})Z_\mu\right] \quad (5.41)$$

For A_μ to be the photon one imposes the conditions

$$\begin{array}{ccccc} \frac{1}{2}(g\sin\theta_W + g'y_\phi\cos\theta_W) = e\,e_1 & & g'y_\phi\cos\theta_W = e(e_1+e_2) & & g'\cos\theta_W = e \\ & \Rightarrow & & \Rightarrow & \\ \frac{1}{2}(-g\sin\theta_W + g'y_\phi\cos\theta_W) = e\,e_2 & & g\sin\theta_W = e(e_1-e_2) = e & & g\sin\theta_W = e, \end{array} \quad (5.42)$$

where the rightmost equalities are a consequence, eq. (4.29), of the Gell-Mann/Nishijima relation. Eliminating g, g', y_ϕ in favour of e, θ_W and the charges one finds

$$-ig\frac{\tau_3}{2}W_{3\mu} - ig'\frac{y_\phi}{2}B_\mu = -ie\begin{pmatrix} e_1A_\mu & 0 \\ 0 & e_2A_\mu \end{pmatrix} - \frac{i\,e}{\sin\theta_W\cos\theta_W}\begin{pmatrix} \frac{1}{2} - e_1\sin^2\theta_W Z_\mu & 0 \\ 0 & -\frac{1}{2} - e_2\sin^2\theta_W Z_\mu \end{pmatrix}$$

Going back to the full expression, eq. (5.20), including the $W^\pm_\mu$ contribution, the covariant derivative on a doublet field is

$$\boxed{\begin{aligned} D_\mu = \partial_\mu - i\frac{e}{\sqrt{2}\sin\theta_W}\begin{pmatrix} 0 & W^+_\mu \\ W^-_\mu & 0 \end{pmatrix} - ie\begin{pmatrix} e_1A_\mu & 0 \\ 0 & e_2A_\mu \end{pmatrix} \\ -i\frac{e}{\sin\theta_W\cos\theta_W}\begin{pmatrix} (\frac{1}{2} - e_1\sin^2\theta_W)Z_\mu & 0 \\ 0 & (-\frac{1}{2} - e_2\sin^2\theta_W)Z_\mu \end{pmatrix} \end{aligned}} \quad (5.43)$$

Since, by definition, $W^{-*}_\mu = W^+_\mu$, from now on we use the notation $W^-_\mu = W_\mu$ and $W^+_\mu = W^*_\mu$ to respectively represent the wave functions of the W^- and W^+ gauge bosons.

The covariant derivative acting on a singlet ϕ is simply

$$\boxed{D_\mu = \partial_\mu - ig'\frac{y_\phi}{2}B_\mu = \partial_\mu - ie\,e_\phi A_\mu + i\frac{e\,e_\phi\sin^2\theta_W}{\sin\theta_W\cos\theta_W}Z_\mu} \quad (5.44)$$

5.2 Gauge bosons and their self-interactions

We already identified in eq. (5.24) the free gauge boson pieces $\mathcal{L}_{0G}$ and in eq. (5.25) the interacting terms $\mathcal{L}_{IG}$. We now reformulate these expressions in terms of the "physical" fields W^*_μ, W_μ, Z_μ and

A_μ. For this purpose we rewrite $\mathcal{L}_{0G}$ by doing an integration by part and neglecting, as usual, the terms which are total derivatives, we find

$$\mathcal{L}_{0G} = \frac{1}{2} W_{i\mu}(x) \mathcal{D}^{\mu\nu} W^{i\nu}(x) + \frac{1}{2} B_\mu(x) \mathcal{D}^{\mu\nu} B_\nu(x), \tag{5.45}$$

with $\mathcal{D}^{\mu\nu} = \Box g^{\mu\nu} - \partial^\mu \partial^\nu$. This is rewritten in a matrix form

$$\mathcal{L}_{0G} = \frac{1}{2}(W_{1\mu}\ W_{2\mu}) \begin{pmatrix} \mathcal{D}^{\mu\nu} & 0 \\ 0 & \mathcal{D}^{\mu\nu} \end{pmatrix} \begin{pmatrix} W_{1\mu} \\ W_{2\mu} \end{pmatrix} + \frac{1}{2}(W_{3\mu}\ B_\mu) \begin{pmatrix} \mathcal{D}^{\mu\nu} & 0 \\ 0 & \mathcal{D}^{\mu\nu} \end{pmatrix} \begin{pmatrix} W_{3\mu} \\ B_\mu \end{pmatrix}. \tag{5.46}$$

We go from the $W_{3\mu}, B_\mu$ coordinates to the A_μ, Z_μ coordinates by a rotation matrix $\mathcal{R}$, eq. (5.31), and since $\mathcal{R}^T\mathcal{R} = 1$, we can immediately replace $(W_{3\mu}\ B_\mu)$ by $(A_\mu\ Z_\mu)$ in the equation above. Now we go from the $W_{1\mu}, W_{2\mu}$ components to the charged W's ones via the matrix $\mathcal{O}$ defined by

$$\begin{pmatrix} W_{1\mu} \\ W_{2\mu} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} W^*_\mu \\ W_\mu \end{pmatrix}, \tag{5.47}$$

which satisfies $\mathcal{O}^T\mathcal{O} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so that we can immediately write

$$\begin{aligned} \mathcal{L}_{0G} &= \frac{1}{2}[W^*_\mu(x)\mathcal{D}^{\mu\nu}W_\nu(x) + W_\mu(x)\mathcal{D}^{\mu\nu}W^*_\nu(x)] + \frac{1}{2}Z_\mu(x)\mathcal{D}^{\mu\nu}Z_\nu(x) + \frac{1}{2}A_\mu(x)\mathcal{D}^{\mu\nu}A_\nu(x) \\ &= -\frac{1}{4}\mathcal{K}^*_{\mu\nu}\mathcal{K}^{\mu\nu} - \frac{1}{4}\mathcal{K}_{\mu\nu}\mathcal{K}^{*\mu\nu} - \frac{1}{4}\mathcal{K}_{Z\mu\nu}\mathcal{K}^{\mu\nu}_Z - \frac{1}{4}\mathcal{K}_{A\mu\nu}\mathcal{K}^{\mu\nu}_A, \end{aligned} \tag{5.48}$$

where in the last line we have dropped a total derivative and where the $\mathcal{K}^{*\mu\nu}, \mathcal{K}^{\mu\nu}, \mathcal{K}^{\mu\nu}_Z, \mathcal{K}^{\mu\nu}_A$ are respectively the abelian-like stress-energy tensors, eq. (5.21), of the $W^\pm, Z, A$ gauge bosons.

We turn now to the interaction lagrangian density $\mathcal{L}_{IG}$ eq. (5.25). Permuting $\mu \leftrightarrow \nu, j \leftrightarrow k$ in the term $\epsilon_{ijk}\partial_\nu W_{i\mu}(x)W^\mu_j(x)W^\nu_k(x)$ one obtains

$$\mathcal{L}_{IG} = -g\ \epsilon_{ijk}\partial_\mu W_{i\nu}(x)W^\mu_j(x)W^\nu_k(x) - \frac{g^2}{4}\epsilon_{ijk}W_{j\mu}(x)W_{k\nu}(x)\ \epsilon_{ilm}W^\mu_l(x)W^\nu_m(x). \tag{5.49}$$

The term linear in g can be written

$$-\,g\ \det \begin{vmatrix} \partial_\mu W_{1\nu} & W^\mu_1 & W^\nu_1 \\ \partial_\mu W_{2\nu} & W^\mu_2 & W^\nu_2 \\ \partial_\mu W_{3\nu} & W^\mu_3 & W^\nu_3 \end{vmatrix}. \tag{5.50}$$

Adding $i\times$the second line to te first one to reconstruct W^*_μ and taking into account the fact that a determinant is invariant when adding or subtracting lines (eventually multiplied by a constant factor) one obtains for the expression (5.50)

$$
\begin{aligned}
-g\,\det\begin{vmatrix} \sqrt{2}\partial_\mu W^*_\nu & \sqrt{2}W^{*\mu} & \sqrt{2}W^{*\nu} \\ \partial_\mu W_{2\nu} & W_2^\mu & W_2^\nu \\ \partial_\mu W_{3\nu} & W_3^\mu & W_3^\nu \end{vmatrix} &= -\frac{g}{2\,i}\det\begin{vmatrix} \sqrt{2}\partial_\mu W^*_\nu & \sqrt{2}W^{*\mu} & \sqrt{2}W^{*\nu} \\ 2\,i\partial_\mu W_{2\nu} & 2\,iW_2^\mu & 2\,iW_2^\nu \\ \partial_\mu W_{3\nu} & W_3^\mu & W_3^\nu \end{vmatrix} \\
&= i\,g\det\begin{vmatrix} \partial_\mu W^*_\nu & W^{*\mu} & W^{*\nu} \\ \partial_\mu W_\nu & W^\mu & W^\nu \\ \partial_\mu W_{3\nu} & W_3^\mu & W_3^\nu \end{vmatrix}. \qquad (5.51)
\end{aligned}
$$

The last equality is obtained by subtracting the first line from the second. Then using $W_{3\nu} = \sin\theta_W A_\mu + \cos\theta_W Z_\mu$ and the relation $e = g\sin\theta_W$ (eq. (5.34)), the above expression becomes

$$
i\,e\,\det\begin{vmatrix} \partial_\mu W^*_\nu & W^{*\mu} & W^{*\nu} \\ \partial_\mu W_\nu & W^\mu & W^\nu \\ \partial_\mu A_\nu & A^\mu & A^\nu \end{vmatrix} + i\,e\,\frac{\cos\theta_W}{\sin\theta_W}\det\begin{vmatrix} \partial_\mu W^*_\nu & W^{*\mu} & W^{*\nu} \\ \partial_\mu W_\nu & W^\mu & W^\nu \\ \partial_\mu Z_\nu & Z^\mu & Z^\nu \end{vmatrix}. \qquad (5.52)
$$

Expanding the determinant we find for the γW^+W^- vertex

$$
-\,i\,e\,[\partial_\mu W^*_\nu(W^\mu A^\nu - A^\mu W^\nu) - \partial_\mu W_\nu(W^{*\mu}A^\nu - A^\mu W^{*\nu}) + \partial_\mu A_\nu(W^{*\mu}W^\nu - W^{-\mu}W^{*\nu})], \qquad (5.53)
$$

By assigning a definite index to each field, *e.g.* $A^\lambda, W^\rho, W^{*\sigma}$, the expression takes the usual form

$$
i\,e\,[A^\lambda g^{\rho\sigma}(W_\rho\partial_\lambda W^*_\sigma - W^*_\sigma\partial_\lambda W_\rho) + W^\rho g^{\sigma\lambda}(W^*_\sigma\partial_\rho A_\lambda - A_\lambda\partial_\rho W^*_\sigma) + W^{*\sigma}g^{\lambda\rho}(A_\lambda\partial_\sigma W_\rho - W_\rho\partial_\sigma A_\lambda)], \quad (5.54)
$$

and similarly for the ZW^+W^- vertex with the coupling $e\cos\theta_W/\sin\theta_W$ instead of e. This defines all tri-linear couplings among gauge bosons.

The term in g^2 in the interaction lagrangian density eq. (5.49) is rather boring to expand. Using the relation $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$, it becomes

$$
\begin{aligned}
&-\frac{e^2}{4\sin^2\theta_W}\,[\mathbf{W}^\mu(x)\cdot\mathbf{W}_\mu(x)\mathbf{W}^\nu(x)\cdot\mathbf{W}_\nu(x) - \mathbf{W}^\mu(x)\cdot\mathbf{W}_\nu(x)\mathbf{W}^\nu(x)\cdot\mathbf{W}_\mu(x)] \\
&= -\frac{e^2}{4\sin^2\theta_W}\left[W_i^\mu\,W_{i\rho}\,W_j^\nu\,W_{j\sigma}\right]\,[g_\mu^\rho\,g_\nu^\sigma - g_\mu^\sigma\,g_\nu^\rho] \qquad (5.55)
\end{aligned}
$$

with the notation $\mathbf{W}^\mu\cdot\mathbf{W}_\nu = \Sigma_i W_i^\mu W_{i\nu}$. One obtains the vertex for the physical fields using

$$
\mathbf{W}^\mu(x)\cdot\mathbf{W}_\rho(x) = W^\mu W^*_\rho + W^{*\mu}W_\rho + (\sin\theta_W A^\mu + \cos\theta_W Z^\mu)(\sin\theta_W A_\rho + \cos\theta_W Z_\rho)], \qquad (5.56)
$$

so that eq. (5.55) becomes

$$
-\frac{e^2}{2\sin^2\theta_W}\left[W_\mu W^*_\rho W_\sigma W^*_\nu + W_\mu W^*_\rho(\sin\theta_W A_\sigma + \cos\theta_W Z_\sigma)(\sin\theta_W A_\nu + \cos\theta_W Z_\nu)\right][g^{\mu\rho}\,g^{\nu\sigma} - g^{\mu\sigma}\,g^{\nu\rho}]
$$

The antisymmetry of the $[g^{\mu\rho}\, g^{\nu\sigma} - g^{\mu\sigma}\, g^{\nu\rho}]$ tensor combination kills the terms with only photons and/or Z bosons. The self-couplings of gauge bosons are thus given by

$$\begin{aligned}
\mathcal{L}_{IG} = & \\
& -i\, e\, [A^{\lambda} g^{\rho\sigma} (W_\rho \partial_\lambda W^*_\sigma - W^*_\sigma \partial_\lambda W_\rho) + W^\rho g^{\sigma\lambda} (W^*_\sigma \partial_\rho A_\lambda - A_\lambda \partial_\rho W^*_\sigma) + W^{*\sigma} g^{\lambda\rho} (A_\lambda \partial_\sigma W_\rho - W_\rho \partial_\sigma A_\lambda)] \\
& + \{A_\lambda \to Z_\lambda, \quad e \to e \cos\theta_W / \sin\theta_W\} \qquad (5.57) \\
& - \frac{e^2}{2\sin^2\theta_W} \left[W_\mu W^*_\rho W_\sigma W^*_\nu + W_\mu W^*_\rho (\sin\theta_W A_\sigma + \cos\theta_W Z_\sigma)(\sin\theta_W A_\nu + \cos\theta_W Z_\nu)\right] [g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}]
\end{aligned}$$

In conclusion, from eq. (5.49) one has two three-boson vertices $W^- W^+ \gamma$, $W^- W^+ Z$ with derivative couplings and four four-boson vertices $W^- W^+ W^- W^+$, $W^- W^+ \gamma\gamma$, $W^- W^+ ZZ$, $W^- W^+ \gamma Z$. The absence of vertices involving only γ's and/or Z's has its origin in the fact that they would arise from the term $g^2 \epsilon_{i33} W_{3\mu}(x) W_{3\nu}(x)\ \epsilon_{i33} W_3^\rho(x) W_3^\sigma(x)$, in eq. (5.49), which is of course 0. Using "standard methods" one can, from the expressions above, extract the Feynman rules for the couplings between fermions and gauge bosons. It will not be done here as they can be found in books.

To summarize this rather technical section we count at this point 15 couplings in the model (for one generation of fermions). One has:
- 9 fermion-fermion-boson vertices: $\bar{e}e\gamma$, $\bar{e}eZ$, $\bar{\nu}_e\nu_e Z$, $\bar{\nu}_e e W^+$, $\bar{u}u\gamma$, $\bar{u}uZ$, $\bar{d}d\gamma$, $\bar{d}dZ$, $\bar{u}dW^+$
- 2 trilinear gauge bosons vertices : $W^+W^-\gamma$, W^+W^-Z
- 4 quadrilinear gauge bosons vertices : $W^-W^+W^-W^+$, $W^-W^+\gamma\gamma$, W^-W^+ZZ, $W^-W^+\gamma Z$.

They depend only on two parameters e and θ_W (and, of course, the fermion charges). It is obvious that the symmetry properties of the lagrangien is quite constraining. The important fact is that the relations between couplings derived above will be preserved by the mechanism of "spontaneous symmetry breaking" we are going to discuss. This is an important difference with a mechanism of explicit symmetry breaking where these relations would have been lost.

5.3 Progress status and problems

Considering what has been achieved until now, one finds that the model based on the $SU(2)_L \otimes U(1)_Y$ symmetry contains four gauge bosons: two charged ones with $(V - A)$ couplings to fermions and two neutral ones with couplings such that these bosons can be interpreted as the photon and the Z boson. The "only" difference with the real world is that in the present state of development of the model the gauge bosons are massless, because of the assumed exact gauge invariance and the fermions are also massless because of the left-right asymmetry of the gauge group. Counting the bosonic degrees of freedom of the model one realizes that three degrees of freedom are "missing", associated to the longitudinal polarisation states of the heavy vector bosons as summarised in the table.

	Model			Real World	
	degrees of freedom			degrees of freedom	
	transverse	longitudinal		transverse	longitudinal
W^-	2	0	W^-	2	1
W^+	2	0	W^+	2	1
Z	2	0	Z	2	1
γ	2	0	γ	2	0

In order to complete the model one should therefore introduce at least three new fields in the lagrangian. This will be done through a multiplet of scalar fields and it will be seen that, by the mechanism of spontaneous symmetry breaking of local gauge invariance, some of the scalar fields become the longitudinal polarisation states and correlatively the vector bosons acquire a mass.