

6 Spontaneous symmetry breaking under a global phase change

We proceed in steps and discuss, first, the case of a global symmetry and state the Goldstone theorem. In the next sections we deal with the case of a spontaneously broken local $U(1)$ symmetry, leading to a massive gauge boson, and then we turn to the Glashow-Weinberg-Salam model based on a broken $SU(2)_L \otimes U(1)_Y$ symmetry.

6.1 Global symmetry breaking

Consider the very simple case of a complex scalar field

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) \quad (6.1)$$

which has two degrees of freedom $\varphi_1(x), \varphi_2(x)$. The lagrangian

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - V(\varphi) \text{ with the potential } V(\varphi) = -\mu^2 |\varphi|^2 + h |\varphi|^4. \quad (6.2)$$

is invariant under a rigid $U(1)$ phase transformation $\varphi(x) \rightarrow e^{i\alpha} \varphi(x)$ where α is constant. The potential has the well-known “Mexican hat” or “cul-de-bouteille” shape (depending on your cultural background!). The hamiltonian is

$$\begin{aligned} H &= \pi \partial_0 \varphi - \mathcal{L}, \text{ with } \pi = \frac{\delta \mathcal{L}}{\delta \partial_0 \varphi} = \partial_0 \varphi^* \\ &= \underbrace{|\vec{\nabla} \varphi|^2}_{H_{\text{kinetic}}} + V(\varphi). \end{aligned} \quad (6.3)$$

The (positive) kinetic part vanishes for static configurations and the full hamiltonian is minimal for constant values of the field given by

$$|\varphi_0| = \frac{\mu}{\sqrt{2h}} = \frac{v}{\sqrt{2}} \quad (6.4)$$

which defines the so-called vacuum expectation value v of the field φ in terms of the parameters of the lagrangian. Indeed, the quantum theory should be constructed from the lowest energy classical state which, in this case, is characterised by having its norm constrained by the above equation. One immediately notices that the vacuum is degenerate since the application of a gauge transformation (phase change) does not affect the norm of the state. There is an infinite number of classical vacuum states, namely all states of type $|\varphi_0| e^{i\alpha}$. However to construct the quantum theory one needs to choose a particular vacuum, by imposing, for example, the classical vacuum field to be real *i.e.*

$$\varphi_0 = \frac{v}{\sqrt{2}} \quad (6.5)$$

This obviously amounts to breaking the symmetry of the vacuum since φ_0 is no more invariant under a gauge transformation, but the dynamical laws are still unbroken because they are given by the gauge invariant lagrangian eq. (6.2). This is the basis of “spontaneous symmetry breaking” in contradistinction to “explicit symmetry breaking” where the lagrangian itself would loose gauge invariance. To study the theory, we translate the original field by its vacuum expectation value

$$\varphi(x) = \frac{1}{\sqrt{2}}(v + \varphi_1(x) + i\varphi_2(x)) \quad (6.6)$$

and, neglecting constant terms, the lagrangian becomes

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_\mu \varphi_1)^2 - hv^2 \varphi_1^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 \\ & - hv\varphi_1(\varphi_1^2 + \varphi_2^2) - \frac{h}{4}(\varphi_1^2 + \varphi_2^2)^2 \end{aligned} \quad (6.7)$$

After spontaneous symmetry breaking, we are left with a model of two interacting real fields φ_1 and φ_2 . The free theory is given by the first line of the equation above which shows that φ_1 has a mass $m_{\varphi_1} = \sqrt{2}hv$ while φ_2 is massless: φ_2 is called the Goldstone boson. The interaction part is all contained in the second line of eq. (6.7) and there are cubic and quartic interactions between φ_1 and φ_2 . Since, the initial lagrangian contained only two parameters, there are necessarily relations between the three parameters m_{φ_1} , the coefficient of the cubic coupling term g_3 and the coefficient of the quartic coupling g_4 *e.g.*

$$g_3^2 = 2m_{\varphi_1}^2 g_4. \quad (6.8)$$

Such a relation reflects the symmetry property of the lagrangian density. These features are a simple illustration of very general properties of spontaneous breaking of larger (non-abelian) group symmetry. They are a particular case of the Goldstone theorem.

6.2 The Goldstone theorem

This theorem reads :

When a global symmetry is spontaneously broken there appear as many massless scalar modes (called the Goldstone bosons) as there are broken degrees of symmetry.

A proof of this theorem is now sketched. Consider φ , a collection of n scalar fields φ_i , $i = 1, \dots, n$ written as a column vector so that

$$\varphi^T = (\varphi_1, \dots, \varphi_n) \quad (6.9)$$

The lagrangian density is formally written as

$$\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)_{\text{kin}} - V(\varphi). \quad (6.10)$$

The vacuum of the model is defined by the conditions

$$\frac{\delta V}{\delta \varphi_i} = 0, \Rightarrow \text{vacuum: } \varphi^{0T} = (\varphi_1^0, \dots, \varphi_n^0) \quad (6.11)$$

One perturbs around the vacuum state

$$\varphi = \varphi^0 + \varphi', \quad \text{i.e. } \varphi_i = \varphi_i^0 + \varphi'_i \quad (6.12)$$

so that the lagrangian (neglecting constant terms) is re-written

$$\mathcal{L} = \mathcal{L}(\varphi', \partial_\mu \varphi')_{\text{kin}} - \frac{1}{2} \sum_{ij} \frac{\delta V}{\delta \varphi_i \delta \varphi_j} \Big|_{\varphi^0} \varphi'_i \varphi'_j \oplus (\varphi'^3) \oplus (\varphi'^4) \quad (6.13)$$

where it is not necessary for our present purposes to specify the cubic nor the quartic couplings. By construction, there are no terms linear in the fields because we are expanding around the minimum of the potential. The quantity of interest is the quadratic term which defines the mass matrix

$$m_{ij}^2 = \frac{\delta V}{\delta \varphi_i \delta \varphi_j} \Big|_{\varphi^0}. \quad (6.14)$$

Consider now the action of an infinitesimal global gauge transformation. Its action on the fields is

$$\delta \varphi = i \alpha^J T^J \varphi, \quad J = 1, \dots, N, \quad (6.15)$$

where the T^J are the N generators ($n \times n$ matrices) of the group and the α^J are the N associated arbitrary parameters. If for some field configuration φ we have for a particular generator T^J ,

$$T^J \varphi = 0, \Rightarrow \delta \varphi = i \alpha^J T^J \varphi = 0, \quad (6.16)$$

then we say that this configuration φ is invariant under the sub-group generated by T^J : the corresponding symmetry is unbroken. If, on the contrary, $T^J \varphi \neq 0$ the corresponding degree of symmetry is said to be spontaneously broken. Let us suppose now that the vacuum satisfies

$$\begin{aligned} T^J \varphi^0 &\neq 0 & \text{for } J = 1, \dots, N' \\ T^J \varphi^0 &= 0 & \text{for } J = N' + 1, \dots, N, \end{aligned} \quad (6.17)$$

i.e. that the vacuum state breaks N' degrees of symmetry. The invariance of the potential $V(\varphi)$ under the gauge transformation $\delta \varphi_i = i \alpha^J T_{ik}^J \varphi_k$ yields

$$\delta V(\varphi) = \frac{\delta V}{\delta \varphi_i} \delta \varphi_i = i \alpha^J \frac{\delta V}{\delta \varphi_i} T_{ik}^J \varphi_k = 0. \quad (6.18)$$

Since this is true for any α^J one has

$$\frac{\delta V}{\delta \varphi_i} T_{ik}^J \varphi_k = 0. \quad (6.19)$$

Taking the derivative of this relation at $\varphi = \varphi^0$, it comes out

$$\left. \frac{\delta^2 V}{\delta \varphi_j \delta \varphi_i} \right|_{\varphi^0} T_{ik}^J \varphi_k^0 + \left. \frac{\delta V}{\delta \varphi_i} \right|_{\varphi^0} T_{ik}^J \delta_{kj} = 0 \Rightarrow m_{ji}^2 T_{ik}^J \varphi_k^0 = 0, \quad (6.20)$$

where the last equality is true because φ^0 defines the minimum of the potential. Since this relation is automatically satisfied for $J = N' + 1, \dots, N$ one concludes that the mass matrix must have N' vanishing eigenvalues. Thus, N' fields φ'_i will be massless which are the Goldstone bosons associated to the N' degrees of broken symmetry (*qed*).