7 Spontaneous local U(1) symmetry breaking

We impose now that the lagrangian density eq. (6.2) is invariant under the local phase change $\varphi(x) \to \mathrm{e}^{ig\alpha(x)}\varphi(x)$. For this purpose we introduce a vector field $B_\mu(x)$ and a covariant derivative $D_\mu = \partial_\mu - igB_\mu(x)$ such that $D_\mu\varphi(x) \to ig\alpha(x)D_\mu\varphi(x)$ under an infinitesimal phase change. This is realised if $B_\mu(x)$ transforms as $B_\mu(x) \to B_\mu(x) + g\partial_\mu\alpha(x)$. Since $D_\mu\varphi^*(x) \to -ig\alpha(x)D_\mu\varphi^*(x)$ the locally invariant version of the scalar field lagrangian density is

$$\mathcal{L}_S + \mathcal{L}_G = D_\mu\varphi^* D_\mu\varphi + \mu^2\varphi^*\varphi - h(\varphi^*\varphi)^2 - \frac{1}{4}\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} \tag{7.1}$$

where we have also included the kinetic term, see eq. (5.21), of the gauge boson $B_\mu(x)$. As in the study of the breaking of the global symmetry we choose as the lowest energy state $\varphi_0 = v/\sqrt{2}$, (eq. (6.5), and we expand the field around this vacuum expectation value as in eq. (6.6).

7.1 Unitary gauge

We take advantage of the freedom of choice of the gauge to find a function $\alpha(x)$ such that $\mathrm{e}^{ig\alpha(x)}$ applied to eq. (6.6) gives

$$\varphi(x) = \frac{1}{\sqrt{2}}(v + H(x)), \tag{7.2}$$

i.e. we absorb the imaginary part in a change of phase and we are left with one real field $H(x)$. This choice defines the unitary gauge. Applying the covariant derivative on $\varphi(x)$ one obtains

$$D_\mu\varphi(x) = \frac{1}{\sqrt{2}}\partial_\mu H(x) - igB_\mu(x)\frac{1}{\sqrt{2}}(v + H(x)) \tag{7.3}$$

Injecting this in the lagrangian density, taking the potential part from eq. (6.7) with $\varphi_1 = H$, $\varphi_2 = 0$, and reshuffling the terms we find

$$\boxed{\begin{aligned}\mathcal{L}_S + \mathcal{L}_G = &\left[\frac{1}{2}(\partial_\mu H(x))^2 - hv^2H^2(x)\right] + \left[-\frac{1}{4}\mathcal{K}_{\mu\nu}(x)\mathcal{K}^{\mu\nu}(x) + \frac{g^2v^2}{2}B_\mu(x)B^\mu(x)\right]\\ &+ g^2vH(x)B_\mu(x)B^\mu(x) + \frac{g^2}{2}H^2(x)B_\mu(x)B^\mu(x) - hvH^3(x) - \frac{h}{4}H^4(x).\end{aligned}} \tag{7.4}$$

The terms in the first line are those from which we build the propagators of the H and B_μ fields respectively, while the second line contains the couplings between the fields. Applying the Euler-Lagrange equation (4.1) we obtain for the H field ($\partial_\mu\partial^\mu = \Box$)

$$(-\Box - 2hv^2)H(x) = 3hvH^2(x) + hH^3(x) - g^2vB_\mu(x)B^\mu(x) - g^2H(x)B_\mu(x)B^\mu(x), \tag{7.5}$$

and for the gauge boson

$$(\Box g_{\mu\nu} - \partial_\mu \partial_\nu + (gv)^2) B^\nu(x) = -g^2 H^2(x) B_\mu(x) - 2g^2 v H B_\mu(x). \tag{7.6}$$

To get the free propagators one solves the Green's functions

$$\begin{aligned} (-\Box - 2hv^2)\, G(x-y) &= i\delta^{(4)}(x-y) \\ (\Box g_{\mu\rho} - \partial_\mu \partial_\rho + (gv)^2 g_{\mu\rho})\, G^{\rho\nu}(x-y) &= i g_\mu^\nu \delta^{(4)}(x-y), \end{aligned} \tag{7.7}$$

in Fourier space. For the scalar field one parameterises $G(x-y) = \int (d^4k/(2\pi)^4) \exp(-ik(x-y)) G(k)$ and one easily get the H field propagator

$$\boxed{G(k) = \frac{i}{k^2 - M_{_H}^2 + i\epsilon} \qquad \text{with} \quad M_{_H} = v\sqrt{2h},} \tag{7.8}$$

with the $i\epsilon$ prescription required by causality. Similarly, for the gauge field we write $G^{\mu\nu}(x-y) = \int (d^4k/(2\pi)^4) \exp(-ik(x-y)) G^{\mu\nu}(k)$ to get

$$(-k^2 g_{\mu\rho} + k_\mu k_\rho + (gv)^2 g_{\mu\rho}) G^{\rho\nu}(k) = i g_\mu^\nu. \tag{7.9}$$

We look for the solution under the form $G^{\rho\nu}(k) = a g^{\rho\nu} + b k^\rho k^\nu$ which is the most general rank 2 tensor which can be constructed from a vector k^μ. One obtains finally

$$\boxed{G_{\mu\nu}(k) = \frac{-i}{k^2 - M_{_B}^2 + i\epsilon}\, (g_{\mu\nu} - \frac{k_\mu k_\nu}{M_{_B}^2}) \qquad \text{with} \quad M_{_B} = gv.} \tag{7.10}$$

The mass of the scalar H field is $M_{_H} = \sqrt{2h}v$ and the mass of the gauge field $M_{_B} = gv$: both are proportional to the vacuum expectation value of the scalar field but the latter is proportional to the gauge coupling while the former depends on the quartic coupling in the potential. The term giving rise to the gauge boson mass originates from the covariant derivative acting on $\varphi(x)$ after symmetry breaking while the mass of the H field comes from the potential $V(\varphi)$.

• Remark on the polarisation of a massive vector boson

The propagator of $B_\mu(x)$ is that of a massive scalar field which has three states of polarisation. Indeed one can easily verify, from eq. (2.25), that the numerator of eq. (7.10) is

$$-\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_{_B}^2}\right) = \sum_i \varepsilon_\mu^{(i)}(k)\, \varepsilon_\nu^{(i)}(k), \tag{7.11}$$

the trace of which is -3.

Counting the degrees of freedom in the model we have after symmetry breaking one real scalar field $H(x)$ and the three polarisation states of the gauge boson while before symmetry breaking one had two scalar fields $\varphi_1(x), \varphi_2(x)$ and the two polarisation states of the massless gauge boson : it appears that the massless Golstone boson $\varphi_2(x)$ has become the longitudinal polarisation of $B_\mu(x)$. The gauge used in this derivation is called the unitary gauge. With this choice the vector boson propagator may lead, as we have seen, to divergences when calculationg Feynman diagrams because of the $k_\mu k_\nu/m_B^2$ term and therefore may ruin the renormalisability of the model.

7.2 Renormalisable gauges : 't Hooft R_ξ gauges

To study this is more detail we go back to the lagrangiann density eq. (7.1) with the general form, eq. (6.6), of the scalar field after symmetry breaking. The covariant derivative is then

$$\begin{aligned} D_\mu\phi(x) &= (\partial_\mu - igB_\mu(x))\frac{1}{\sqrt{2}}(v+\phi_1(x)+i\phi_2(x)) \\ &= \frac{1}{\sqrt{2}}[\partial_\mu\phi_1(x)+gB_\mu(x)\phi_2(x)] + \frac{i}{\sqrt{2}}[\partial_\mu\phi_2(x) - gB_\mu(x)(v+\phi_1(x))] \end{aligned} \tag{7.12}$$

The lagrangian density takes then the form, keeping explicitely only the terms quadratic in the fields,

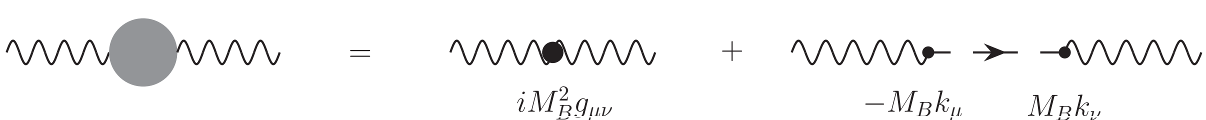
$$\boxed{\begin{aligned} \mathcal{L}_S + \mathcal{L}_G = &\left[\frac{1}{2}(\partial_\mu\phi_1(x))^2 - hv^2\phi_1^2(x)\right] + \left[-\frac{1}{4}\mathcal{K}_{\mu\nu}(x)\mathcal{K}^{\mu\nu}(x) + \frac{g^2v^2}{2}B_\mu(x)B^\mu(x)\right] \\ &+ \left[\frac{1}{2}(\partial_\mu\phi_2(x))^2 - gvB_\mu(x)\partial^\mu\phi_2(x)\right] + \mathcal{L}_{\text{int}}. \end{aligned}} \tag{7.13}$$

The first line is identical to that of eq. (7.4) with a massive scalar field $\phi_1(x)$ ($\phi_1(x) = H(x)$ is the Higgs field) and a massive gauge boson. In the second line one has the massless $\phi_2(x)$ scalar (the Goldstone boson) coupling to the gauge field. The function $\mathcal{L}_{\text{int}}$,

$$\mathcal{L}_{\text{int}} = g(\varphi_2\overleftrightarrow{\partial_\mu}\varphi_1)B^\mu + g^2v\varphi_1B_\mu B^\mu + \frac{g^2}{2}(\varphi_1^2+\varphi_2^2)B_\mu B^\mu - hv\varphi_1(\varphi_1^2+\varphi_2^2) - \frac{h}{4}(\varphi_1^2+\varphi_2^2)^2, \tag{7.14}$$

contains the couplings between ϕ_1, ϕ_2 and B_μ.

Clearly $\phi_2(x)$ is not independent on $B_\mu(x)$ since it oscillates into the gauge boson with a derivative coupling as can be seen from eq. (7.13). In fact if we consider the polarisation tensor of the B_μ field, treating both the mass term and the $B_\mu\partial^\mu\phi_2$ as vertices we find ($gv = M_B$)



$$iM_B^2 g^{\mu\nu} + (-M_B k^\mu)\frac{i}{k^2}M_B k^\nu = iM_B^2(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}) \tag{7.15}$$

which is transverse as it should be. From eq. (2.25) one sees that the tensor structure is equivalent to summing over transverse and longitudinal polarisations of B_μ: this shows that $\phi_2(x)$ builds up the longitudinal polarisation of the originally transverse $B_\mu(x)$ field. One may suspect that iterating the self-energy bubble on the B_μ field propagator will reconstruct the propagator of a massive field. This is disccussed more precisely below.

We follow here a procedure familiar from QED. To quantise QED, it is necessary to break the gauge invariance and this is done by adding to the lagrangian a "gauge fixing" term. Here the gauge fixing term is chosen to be

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial_\mu B^\mu(x) + \xi\ gv\ \phi_2(x))^2. \tag{7.16}$$

This choice (instead of the traditional term $-(\partial_\mu B^\mu(x))^2/2\xi$ of QED) is made to eliminate the mixed term $gvB_\mu(x)\partial^\mu\phi_2(x)$ in the lagrangian. This class of gauge conditions is known under the name of 't Hooft's gauges or R_ξ gauges where ξ is an arbitrary real number. One considers the new lagrangian density $\mathcal{L}_S + \mathcal{L}_G + \mathcal{L}_{GF}$ which then becomes

$$\boxed{\begin{aligned}\mathcal{L}_S + \mathcal{L}_G + \mathcal{L}_{GF} = &\left[\frac{1}{2}(\partial_\mu\phi_1(x))^2 - hv^2\phi_1^2(x)\right] + \frac{1}{2}\left[(\partial_\mu\phi_2(x))^2 - \xi(gv)^2\phi_2^2(x)\right] \\ &+ \left[-\frac{1}{4}\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} + \frac{(gv)^2}{2}B_\mu(x)B^\mu(x)\right] + \frac{1}{2\xi}(\partial_\mu B^\mu(x))^2 + \mathcal{L}_{\rm int}\end{aligned}} \tag{7.17}$$

By the specific choice of the gauge condition the mixed term in $B_\mu\partial^\mu\phi_2$ in $\mathcal{L}_S + \mathcal{L}_G$ combines with the term $\phi_2\partial^\mu B_\mu$ in $\mathcal{L}_{GF}$ to give a total derivative which can be safely ignored in perturbation theory. However the Goldstone boson acquires a mass from the gauge fixing lagrangian density. Following the procedure used when working in the unitary gauge one derives the Green's equation for the fields ϕ_i and B_μ, the solution of which gives the free propagators. Thus one obtains

$$\begin{aligned}(-\Box - 2hv^2)G_{\phi_1}(x-y) &= i\delta^{(4)}(x-y) \\ (-\Box - \xi(gv)^2)G_{\phi_2}(x-y) &= i\delta^{(4)}(x-y) \\ (\Box g_{\mu\nu} - (1-\frac{1}{\xi})\partial_\mu\partial_\nu + (gv)^2 g_{\mu\nu})G^{\nu\rho}(x-y) &= ig_\mu^\rho\delta^{(4)}(x-y).\end{aligned} \tag{7.18}$$

For the scalar fields we obtain easily

for the field $\phi_1 = H$ $$\boxed{G_H(k) = \frac{i}{k^2 - M_H^2 + i\epsilon} \quad \text{with} \quad M_H = v\sqrt{2h}} \tag{7.19}$$

for the Goldstone field ϕ_2 $$\boxed{G_{\phi_2}(k) = \frac{-i}{k^2 - \xi M_B^2 + i\epsilon} \qquad \text{with} \quad M_B = gv}. \tag{7.20}$$

For the gauge fields, introducing $G^{\nu\rho}(x-y) = \int (d^4k/(2\pi)^4) \exp(-ik(x-y)) G^{\nu\rho}(k)$ one has to solve

$$(k^2 g_{\mu\nu} - (1 - \frac{1}{\xi}) k_\mu k_\nu - M_B^2 g_{\mu\nu}) G^{\nu\rho}(k) = -i g_\mu^\rho \tag{7.21}$$

One looks for the solution in the form of $a^{\nu\rho} + b k^\nu k^\rho$and one finds

$$\boxed{G_{\nu\rho}(k) = -\frac{i}{k^2 - M_B^2 + i\epsilon} \left(g_{\nu\rho} - (1-\xi) \frac{k_\nu k_\rho}{k^2 - \xi M_B^2} \right).} \tag{7.22}$$

One observes that for any value of ξ finite all propagators have the right asymptotic behavior *i.e.* they behave like $1/k^2, k^2 \to \infty$ which is a necessary condition for the model to be renormalisable. However both the Goldstone and the gauge boson propagators have a spurious pole at $k^2 - \xi m_B^2$ which should cancel when calculating a physical process. It is interesting to compare the gauge boson propagator in the general 't Hooft gauge with its form in the unitary gauge. One proves easily

$$-\frac{i}{k^2 - M_B^2 + i\epsilon} \left(g_{\nu\rho} - (1-\xi) \frac{k_\nu k_\rho}{k^2 - \xi M_B^2} \right) = -\frac{i}{k^2 - M_B^2 + i\epsilon} \left(g_{\nu\rho} - \frac{k_\nu k_\rho}{M_B^2} \right) - \frac{i}{M_B^2} \frac{k_\nu k_\rho}{k^2 - \xi M_B^2} \tag{7.23}$$

One recognises on the right-hand side the propagator in the unitary gauge, eq. (7.10), plus a term which has the the same pole structure as the Goldstone boson. An exemple will be given later, on how such a cancellation occurs between this extra piece and the Goldstone contribution.

Special choices of ξ can be made:
- $\xi = 0$ (Landau gauge) : the Golstone boson is massless and the gauge boson propagator is transverse *i.e.* $k^\nu G_{\nu\rho} = 0$;
- $\xi = 1$ (Feynman gauge) : the Golstone boson has the same mass as the gauge boson but one looses the transversity property of the gauge boson propagator;
- $\xi \to \infty$: the Goldstone boson does not propagate and one keeps only the physical degrees of freedom in the model: one recovers the unitary gauge already considered.

7.3 Fermion masses

We now include a fermion in our toy model. We assume one massless fermion $\psi(x)$ and impose a local $U(1)$ gauge invariance only on the left-handed component of $\psi(x)$: $\delta\psi_L(x) = ig\alpha(x)\psi_L(x)$, $\delta\psi_R(x) = 0$. The fermion part of the lagrangian density takes the form

$$\mathcal{L}_F = \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{\partial} \psi_R, \tag{7.24}$$

with the covariant derivative acting on $\psi_L(x)$ defined by

$$\not{D}\psi_L = (\not{\partial} - ig\not{B})\psi_L. \tag{7.25}$$

We parameterise the $U(1)$ invariant interaction between the scalar field and the fermion by the Yukawa type lagrangian density

$$\mathcal{L}_Y = -\lambda_f(\bar{\psi}_L \phi \psi_R + \bar{\psi}_R \phi^* \psi_L). \qquad (7.26)$$

After symmetry breaking, using the parameterisation eq. (6.6) of the scalar field, only $\mathcal{L}_Y$ is affected

$$\begin{aligned}\mathcal{L}_Y &= -\frac{\lambda_f}{\sqrt{2}}(v+H)(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) - i\frac{\lambda_f}{\sqrt{2}}\phi_2(\bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L) \\ &= -\frac{\lambda_f v}{\sqrt{2}}\bar{\psi}\psi - \frac{\lambda_f}{\sqrt{2}}H\bar{\psi}\psi - i\frac{\lambda_f}{\sqrt{2}}\phi_2\bar{\psi}\gamma^5\psi, \end{aligned} \qquad (7.27)$$

where we have recombined the left-handed and right-handed fields. Regrouping all fermion terms we have

$$\boxed{\mathcal{L}_F + \mathcal{L}_Y = \bar{\psi}(i\partial\!\!\!/ - \frac{\lambda_f v}{\sqrt{2}})\psi + \frac{g}{2}\bar{\psi}B\!\!\!/(1-\gamma^5)\psi - \frac{\lambda_f}{\sqrt{2}}H\bar{\psi}\psi - i\frac{\lambda_f}{\sqrt{2}}\phi_2\bar{\psi}\gamma^5\psi.} \qquad (7.28)$$

We read off the fermion mass

$$\boxed{m_f = \frac{\lambda_f v}{\sqrt{2}}} \qquad (7.29)$$

and the couplings of the fermion
- to the gauge field : $-i(g/2)\gamma_\mu(1-\gamma^5)$;
- to the Higgs field : $i\lambda_f/\sqrt{2})$;
- to the Goldstone boson : $(\lambda_f/\sqrt{2})\gamma^5$.

The coupling of the Higgs to the fermion can be written in terms of "physical parameters", masses and the gauge coupling, and one finds

$$\boxed{\text{coupling Higgs-fermion-fermion : } i\lambda_f/\sqrt{2}) = i\,g\,\frac{m_f}{M_B},} \qquad (7.30)$$

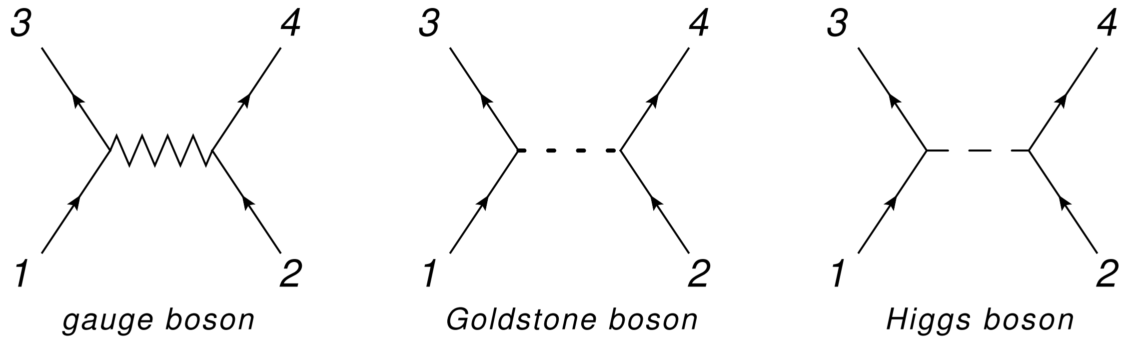
which illustrates an important feature of spontaneous symmetry breaking, namely that the coupling is proportional to the fermion mass.
In the unitary gauge, the Higgs and gauge boson couplings to the fermion are as above, while the Goldstone boson ϕ_2 is absorbed by the gauge choice and does not couple to the fermion.

7.4 Gauge invariance at the Born level: an exemple

Putting everything together, the lagrangien density of our model in a general R_ξ gauge is

$$\mathcal{L}_S + \mathcal{L}_G + \mathcal{L}_{GF} + \mathcal{L}_F + \mathcal{L}_Y \qquad (7.31)$$

with $\mathcal{L}_S + \mathcal{L}_G + \mathcal{L}_{GF}$ from eq. (7.17) and $\mathcal{L}_F + \mathcal{L}_Y$ from eq. (7.28). We are now in a position to calculate the scattering amplitude for the collision $\psi_1 + \psi_2 \to \psi_3 + \psi_4$. The diagrams to be considered are



The point is to check that the first two diagrams lead to a gauge independent contribution since the Higgs exchange diagram is independent of the gauge choice ξ. Using the decomposition eq. (7.23) of the gauge boson propagator in the general 't Hooft gauge it is enough to prove that the rightmost term in eq. (7.23) is cancelled by the Goldstone exchange diagram. From the gauge boson exchange we have

$$-\frac{g^2}{4}\bar{\psi}_3\gamma_\mu(1-\gamma^5)\psi_1(-i)\frac{k^\mu k^\nu}{M_B^2(k^2-\xi M_B^2)}\bar{\psi}_4\gamma_\nu(1-\gamma^5)\psi_2. \tag{7.32}$$

Using Dirac equation this term can be considerably simplified. For instance with $k^\mu = p_1^\mu - p_3^\mu$

$$\begin{aligned}\bar{\psi}_3\gamma_\mu(1-\gamma^5)\psi_1 k^\mu &= \bar{\psi}_3(\not{p}_1-\not{p}_3)\psi_1+\bar{\psi}_3\gamma^5\not{p}_1\psi_1+\bar{\psi}_3\not{p}_3\gamma^5\psi_1\\ &= 2\,m_f\,\bar{\psi}_3\gamma^5\psi_1,\end{aligned} \tag{7.33}$$

where to obtain the last line we have used Dirac equation $\not{p}_1\psi_1 = m_f\psi_1$ and $\bar{\psi}_3\not{p}_3 = m_f\psi_3$. The same trick can be used at the other vertex to obtain

$$\begin{aligned}\bar{\psi}_4\gamma_\mu(1-\gamma^5)\psi_2 k^\mu &= \bar{\psi}_4(\not{p}_4-\not{p}_2)\psi_2-\bar{\psi}_4\not{p}_4\gamma^5\psi_2-\bar{\psi}_4\gamma^5\not{p}_2\psi_2\\ &= -2\,m_f\,\bar{\psi}_4\gamma^5\psi_2.\end{aligned} \tag{7.34}$$

This shows that after symmetry breaking the axial current $\bar{\psi}\gamma_\mu\gamma_5\psi$ is not conserved since, when contracted with the gauge field momentum, it gives a term proportional to the mass of the fermion. Thus eq. (7.32) reduces to

$$-\frac{g^2m_f^2}{M_B^2}\frac{i}{k^2-\xi M_B^2}\bar{\psi}_3\gamma^5\psi_1\bar{\psi}_4\gamma^5\psi_2 \tag{7.35}$$

The contribution of the Goldstone boson exchange is simply

$$\frac{\lambda_F^2}{2}\frac{i}{k^2-\xi M_B^2}\bar{\psi}_3\gamma^5\psi_1\bar{\psi}_4\gamma^5\psi_2 \tag{7.36}$$

Using the relation $g^2m_f^2/M_B^2 = \lambda_F^2/2$, eq. (7.30), one easily verifies the compensation of the ξ dependant part of the gauge propagator by the Goldstone boson. Needless to say that, for this to occur, the mass term of the gauge boson and that of the fermion should have the same origin and be both related to the vacuum expectation value v. The gauge invariance can be checked on other processes notably those involving the triple gauge couplings, however the discussion is more tricky since it implies the coupling of the Goldstone field to the vector boson as given in (7.14).