9 Exercise : study of the reaction proton + proton $\rightarrow$ H + X

The mass of the Higgs boson is large enough to justify the use of the parton model and perturbative QCD to study the production of a Higgs boson in proton-proton collisions.

9.1 The gluon-gluon fusion mechanism

In this framework, considering only the dominant process via gluon-gluon fusion, the hadronic cross section of the inclusive reaction $p(k_1) + p(k_2) \rightarrow H(p_3) + X$ can be written as:

$$\sigma_H = \int_0^1 dx_1 \int_0^1 dx_2\, F_g^P(x_1, M^2)\, F_g^P(x_2, M^2)\, \hat{\sigma}_{g\,g \rightarrow H}, \tag{9.1}$$

where $F_g^P(x, M^2)$ stands for the gluon density in the proton, the gluon carrying a fraction x of the proton four-momentum, evolved at the factorisation scale M. The quantity $\hat{\sigma}_{g\,g\rightarrow H}$ is the cross section of the partonic reaction $g(p_1) + g(p_2) \rightarrow H(p_3)$. The 4-momentum of the initial gluons are such that $p_1 = x_1\, k_1$ and $p_2 = x_2\, k_2$. The partonic cross section itself is given by:

$$\hat{\sigma}_{g\,g\rightarrow H} \quad = \quad \frac{1}{4\, p_1.p_2} \int \frac{d^3 p_3}{(2\,\pi)^3\, 2\, E_3}\, (2\,\pi)^4\, \delta^4(p_1 + p_2 - p_3)\, |\bar{T}|^2, \tag{9.2}$$

with $|\bar{T}|^2$ the matrix element squared averaged over initial polarisations and colours. Transforming $d^3p_3/(2\,E_3)$ in $d^4p_3\, \delta^+(p_3^2 - M_H^2)$, the integration on p_3 can be performed easily with the Dirac distribution and we get:

$$\hat{\sigma}_{g\,g\rightarrow H} \quad = \quad \frac{1}{2\, x_1\, x_2\, S}\, (2\,\pi)\, \delta^+(x_1\, x_2\, S - M_H^2)\, |\bar{T}|^2, \tag{9.3}$$

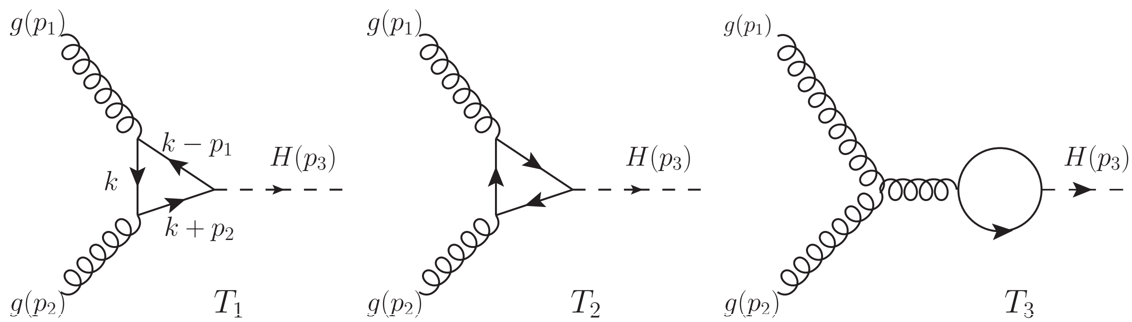
with the total energy squared $S = (k_1 + k_2)^2 = 2\, k_1.k_2$ (the proton mass is neglected). Injecting eq. (9.3) in eq. (9.1), we get for the hadronic cross section:

$$\sigma_H = \int_0^1 \frac{dx_1}{x_1} \int_0^1 \frac{dx_2}{x_2}\, F_g^P(x_1, M^2)\, F_g^P(x_2, M^2)\, \frac{\pi}{S}\, \delta^+(x_1\, x_2\, S - M_H^2)\, |\bar{T}|^2. \tag{9.4}$$

The integration over x_2 can be performed with the help of the remaining Dirac distribution to find:

$$\sigma_H = \frac{\pi}{M_H^2\, S} \int_{M_H^2/S}^1 \frac{dx_1}{x_1}\, F_g^P(x_1, M^2)\, F_g^P\left(\frac{M_H^2}{x_1\, S}, M^2\right)\, |\bar{T}|^2 \tag{9.5}$$

The bounds on integration are obtained from the constraints $0 < x_1, x_2 \leq 1$ and $M_H^2/S \leq x_1 x_2 \leq 1$ which lead to $M_H^2/(x_1\, S) \leq x_2 \leq 1$ and $M_H^2/S \leq x_1 \leq 1$. We compute now $|\bar{T}|^2$. At the lowest order, three diagrams contribute to the partonic process $g(p_1) + g(p_2) \rightarrow H(p_3)$:



The amplitude T_3 vanishes because it is proportional to $Tr[T^a]$, T^a traceless, a generator of the $SU(3)$ colour algebra. Applying the Feynman rules in $n \neq 4$ dimensions to tame potential ultraviolet divergencies (μ is the arbirary mass introduced when going to n dimensions) and taking into account the factor -1 for a fermion loop, the amplitude T_1 is given by:

$$\begin{aligned}
T_1 &= -g_s^2 \frac{m_q}{v}\, Tr\Big[T^a T^b\Big]\, \mu^{4-n}\!\int \frac{d^n k}{(2\,\pi)^n}\, Tr\left[\gamma^\nu \frac{(\not{k} + m_q)}{k^2 - m_q^2 + i\,\epsilon}\,\gamma^\mu \frac{(\not{k} - \not{p}_1 + m_q)}{(k-p_1)^2 - m_q^2 + i\,\epsilon}\,\frac{(\not{k} + \not{p}_2 + m_q)}{(k+p_2)^2 - m_q^2 + i\,\epsilon}\right] \\
&\qquad \times \epsilon_\mu^a(p_1)\,\epsilon_\nu^b(p_2) \\
&= -\frac{g_s^2\, e\, m_q}{4\,\sin\theta_W\, M_W}\, \mu^{4-n}\, \delta^{ab}\, \epsilon_\mu^a(p_1)\,\epsilon_\nu^b(p_2) \int \frac{d^n k}{(2\,\pi)^n} \\
&\qquad \times \frac{Tr\left[\gamma^\nu\,(\not{k} + m_q)\,\gamma^\mu\,(\not{k} - \not{p}_1 + m_q)\,(\not{k} + \not{p}_2 + m_q)\right]}{(k^2 - m_q^2 + i\,\epsilon)\,((k-p_1)^2 - m_q^2 + i\,\epsilon)\,((k+p_2)^2 - m_q^2 + i\,\epsilon)} \qquad (9.6)
\end{aligned}$$

where $g_s T^a$ is the strong interaction coupling of a gluon of colour a to a quark, m_q/v (see eq. (8.29)), the coupling of the quarks to the Higgs boson. In the second equation the relation $1/v = e/2\,\sin\theta_W\, M_W$, eq. (8.19) is used and $Tr\left[T^a T^b\right] = \delta_{ab}/2$ takes care of the sum on the quark colours in the loop. Setting

$$N^{\mu\,\nu}(k) = Tr\left[\gamma^\nu\,(\not{k} + m_q)\,\gamma^\mu\,(\not{k} - \not{p}_1 + m_q)\,(\not{k} + \not{p}_2 + m_q)\right]$$

and computing the trace on the Dirac matrices as usual, we get:

$$\begin{aligned}
N^{\mu\,\nu}(k) &= 4\,m_q\,\{g^{\mu\,\nu}(k-p_1).(k+p_2) + (k+p_2)^\nu\,(k-p_1)^\mu - (k-p_1)^\nu\,(k+p_2)^\mu + k^\nu\,(k+p_2)^\mu \\
&\quad + (k+p_2)^\nu\,k^\mu - g^{\mu\,\nu}\,k.(k+p_2) + k^\nu\,(k-p_1)^\mu + k^\mu\,(k-p_1)^\nu - g^{\mu\,\nu}\,k.(k-p_1) + m_q^2\,g^{\mu\,\nu}\} \\
&= 4\,m_q\,\left\{g^{\mu\,\nu}\,(m_q^2 - k^2 - p_1.p_2) + 4\,k^\mu\,k^\nu - 2\,k^\nu\,p_1^\mu + 2\,k^\mu\,p_2^\nu + p_1^\nu\,p_2^\mu - p_2^\nu\,p_1^\mu\right\}\ . \qquad (9.7)
\end{aligned}$$

In eq. (9.7), all the terms proportional to p_1^μ and p_2^ν can be dropped because they will vanish after contraction with the gluon polarisation vectors. The quantity $N^{\mu\,\nu}(k)$ becomes:

$$N^{\mu\,\nu}(k) = 4\,m_q\,\left\{g^{\mu\,\nu}\,(m_q^2 - k^2 - p_1.p_2) + 4\,k^\mu\,k^\nu + p_1^\nu\,p_2^\mu\right\}. \qquad (9.8)$$

Then, two Feynman parameters x and y are introduced to linearize the denominator:

$$
\begin{aligned}
T_1 &= 2\,K_{\mu\,\nu} \int_0^1 dy\, y \int_0^1 dx \int \frac{d^n k}{(2\,\pi)^n} N^{\mu\,\nu}(k) \\
&\quad \times \left[(1-y)\,(k^2 - m_q^2 + i\,\epsilon) + x\,y\,((k-p_1)^2 - m_q^2 + i\,\epsilon) + (1-x)\,y\,((k+p_2)^2 - m_q^2 + i\,\epsilon)\right]^{-3} \\
&= 2\,K_{\mu\,\nu} \int_0^1 dy\, y \int_0^1 dx \int \frac{d^n k}{(2\,\pi)^n} N^{\mu\,\nu}(k) \\
&\quad \times \left[(k + (p_2\,(1-x) - p_1\,x)\,y)^2 + 2\,y^2\,x\,(1-x) p_1.p_2 - m_q^2 + i\,\epsilon\right]^{-3}
\end{aligned}
\tag{9.9}
$$

with

$$
K_{\mu\,\nu} = -\frac{g_s^2\,e\,m_q}{4\,\sin\theta_W\,M_W}\,\mu^{4-n}\,\delta^{ab}\,\epsilon^a_\mu(p_1)\,\epsilon^b_\nu(p_2)
$$

We shift the loop four-momentum $k = l - (p_2\,(1-x) - p_1\,x)\,y$. The factor $N^{\mu\,\nu}(k)$ contains terms of the type k^2 and $k^\mu\,k^\nu$ which transform under the shift as:

$$
\begin{aligned}
k^2 &\simeq l^2 - 2\,y^2\,x\,(1-x)\,p_1.p_2 \\
k^\mu\,k^\nu &\simeq g^{\mu\,\nu}/n\,l^2 - y^2\,x\,(1-x)\,p_2^\mu\,p_1^\nu
\end{aligned}
$$

All odd powers of l will vanish after the integration over l, so they have been removed. Eq. (9.8) becomes:

$$
N^{\mu\,\nu}(k) = 4\,m_q \left\{ g^{\mu\,\nu} \left[\left(\frac{4}{n} - 1\right) l^2 + m_q^2 + 2\,p_1.p_2 \left(y^2\,x\,(1-x) - \frac{1}{2}\right)\right] + p_1^\nu\,p_2^\mu\,\left(1 - 4\,y^2\,x\,(1-x)\right)\right\}
\tag{9.10}
$$

The amplitude T_1 is then:

$$
T_1 = 2\,K_{\mu\,\nu} \int_0^1 dy\, y \int_0^1 dx \int \frac{d^n l}{(2\,\pi)^n}\,4\,m_q\,\frac{(A_1\,l^2 + A_2)\,g^{\mu\,\nu} + B\,p_1^\nu\,p_2^\mu}{(l^2 - R^2 + i\,\epsilon)^3}
\tag{9.11}
$$

with:

$$
\begin{aligned}
R^2 &= m_q^2 - 2\,y^2\,x\,(1-x)\,p_1.p_2 \\
A_1 &= 4/n - 1 \\
A_2 &= 2\,m_q^2 - p_1.p_2 - R^2 \\
B &= 1 - 4\,y^2\,x\,(1-x) \quad = \quad 1 - 2(m_q^2 + R^2)/p_1.p_2
\end{aligned}
$$

The integration over the four-momentum l yields the following result[15]:

$$
\begin{aligned}
T_1 = \frac{i}{(4\,\pi)^{n/2}}\,4\,m_q\,K_{\mu\,\nu} \int_0^1 dy\, y \int_0^1 dx \Bigg[&\frac{n}{2}\,\frac{4-n}{n}\,\Gamma\left(2 - \frac{n}{2}\right)\left(R^2 - i\,\epsilon\right)^{-2+n/2} g^{\mu\,\nu} \\
&- \Gamma\left(3 - \frac{n}{2}\right)\left(R^2 - i\,\epsilon\right)^{-3+n/2}\left(A_2\,g^{\mu\,\nu} + B\,p_1^\nu\,p_2^\mu\right)\Bigg]
\end{aligned}
\tag{9.12}
$$

[15]The general formula is:

$$
\begin{aligned}
\int \frac{d^n k}{(2\pi)^n}\quad \frac{k^{2^r}}{[k^2 - R^2 + i\epsilon]^m} &= i\,(R^2 - i\epsilon)^{r-m+\frac{n}{2}}\,\frac{(-1)^{r-m}}{(4\pi)^{\frac{n}{2}}}\,\frac{\Gamma(r+\frac{n}{2})}{\Gamma(\frac{n}{2})}\,\frac{\Gamma(m-r-\frac{n}{2})}{\Gamma(m)} \\
&= i\,\frac{(-1)^{r-m}}{(4\pi)^2}\,\left(\frac{4\pi}{R^2 - i\epsilon}\right)^{\varepsilon} (R^2)^{2+r-m}\,\frac{\Gamma(2+r-\varepsilon)}{\Gamma(2-\varepsilon)}\,\frac{\Gamma(m-r-2+\varepsilon)}{\Gamma(m)}
\end{aligned}
$$

.

The coefficient in front of the ultraviolet divergence $\Gamma\left(2-\frac{n}{2}\right)$ vanishes for $n=4$, more precisely:

$$\frac{n}{2}\,\frac{4-n}{n}\,\Gamma\left(2-\frac{n}{2}\right)=\left(2-\frac{n}{2}\right)\,\Gamma\left(2-\frac{n}{2}\right)=\Gamma\left(3-\frac{n}{2}\right)$$

So, actually, there is no divergence in this amplitude and we can now take safely $n=4$ so that $\Gamma\left(3-\frac{n}{2}\right)$ reduces to 1. In addition, using $p_1.p_2=M_H^2/2$, we get:

$$\begin{aligned} T_1 \;=\;& \frac{i}{(4\,\pi)^2}\,4\,m_q\,K_{\mu\,\nu}\left(g^{\mu\,\nu}-\frac{2\,p_1^\nu\,p_2^\mu}{M_H^2}\right)\int_0^1 dy\,y\int_0^1 dx \\ & \times\left[2+\frac{M_H^2}{2}\left(1-\frac{4\,m_q^2}{M_H^2}\right)\frac{1}{m_q^2-y^2\,x\,(1-x)\,M_H^2-i\,\epsilon}\right] \end{aligned} \tag{9.13}$$

To perform the integration on the Feynman parameters, let us introduce the function:

$$J(z)=\int_0^1 dx\int_0^1 dy\,y\,\frac{1}{1-y^2\,x\,(1-x)/z-i\,\epsilon} \tag{9.14}$$

with $z=m_q^2/M_H^2$ positive. The integration over y can be easily performed to get:

$$\begin{aligned} J(z) &= -\frac{z}{2}\int_0^1 dx\,\frac{1}{x\,(1-x)}\,\ln\left(1-\frac{x\,(1-x)}{z}-i\,\epsilon\right) \\ &= -\frac{z}{2}\int_0^1 dx\,\left[\frac{1}{x}+\frac{1}{(1-x)}\right]\,\ln\left(1-\frac{x\,(1-x)}{z}-i\,\epsilon\right) \\ &= -z\int_0^1\frac{dx}{x}\,\ln\left(1-\frac{x\,(1-x)}{z}-i\,\epsilon\right)\;. \end{aligned} \tag{9.15}$$

The roots of the argument of the logarithm are given by:

$$0<z<\tfrac{1}{4}\qquad x_{1,2}=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{1-4\,z}\pm i\,\epsilon$$

$$z>\tfrac{1}{4}\qquad x_{1,2}=\tfrac{1}{2}\pm\tfrac{i}{2}\sqrt{4\,z-1}\;,$$

so

$$\ln\left(1-\frac{x\,(1-x)}{z}-i\,\epsilon\right)=\ln\left(\frac{1}{z}\right)+\ln(x-x_1)+\ln(x-x_2)\;,$$

but $\ln(1/z)=-\ln(x_1\,x_2)=-\ln(-x_1)-\ln(-x_2)$ because x_1 and x_2 are complex conjugate. The two terms $\ln(x-x_1)$ and $\ln(-x_1)$ can be grouped because the imaginary parts of the two arguments are the same and similarly for the terms in x_2. So we get for $J(z)$:

$$\begin{aligned} J(z) &= -z\left[\int_0^1\frac{dx}{x}\,\ln\left(1-\frac{x}{x_1}\right)+\int_0^1\frac{dx}{x}\,\ln\left(1-\frac{x}{x_2}\right)\right] \\ &= z\left[\mathrm{Li}_2\left(\frac{1}{x_1}\right)+\mathrm{Li}_2\left(\frac{1}{x_2}\right)\right]\;. \end{aligned} \tag{9.16}$$

It can be shown, c.f. sec. 9.3, that $J(z)$ can be written using only the logarihtm function whatever the value of z is:

$$J(z) = -\frac{z}{2} \left\{ \begin{array}{cc} \left(\ln\left(\frac{1-\sqrt{1-4\,z}}{1+\sqrt{1-4\,z}}\right) - i\,\pi\right)^2 & z \leq 1/4 \\ \ln^2\left(\frac{i\,\sqrt{4\,z-1}-1}{i\,\sqrt{4\,z-1}+1}\right) & z > 1/4 \end{array} \right. . \tag{9.17}$$

So the amplitude T_1 is given by:

$$\begin{aligned} T_1 &= \frac{i}{(4\,\pi)^2}\, 4\, m_q\, K_{\mu\,\nu} \left(g^{\mu\,\nu} - \frac{2\, p_1^\nu\, p_2^\mu}{M_H^2}\right) \frac{M_H^2}{2\, m_q^2} \\ &\quad \times \left\{ 2\, \frac{m_q^2}{M_H^2} + \left(1 - 4\, \frac{m_q^2}{M_H^2}\right) J\left(\frac{m_q^2}{M_H^2}\right) \right\} . \end{aligned} \tag{9.18}$$

For the following, we set:

$$\mathcal{F}(z) = 2\,z + (1 - 4\,z)\; J\,(z) \tag{9.19}$$

The function $\mathcal{F}(z)$ can be complex or real following the ratio $z = m_q^2/M_H^2$. In fig. 7, we draw the

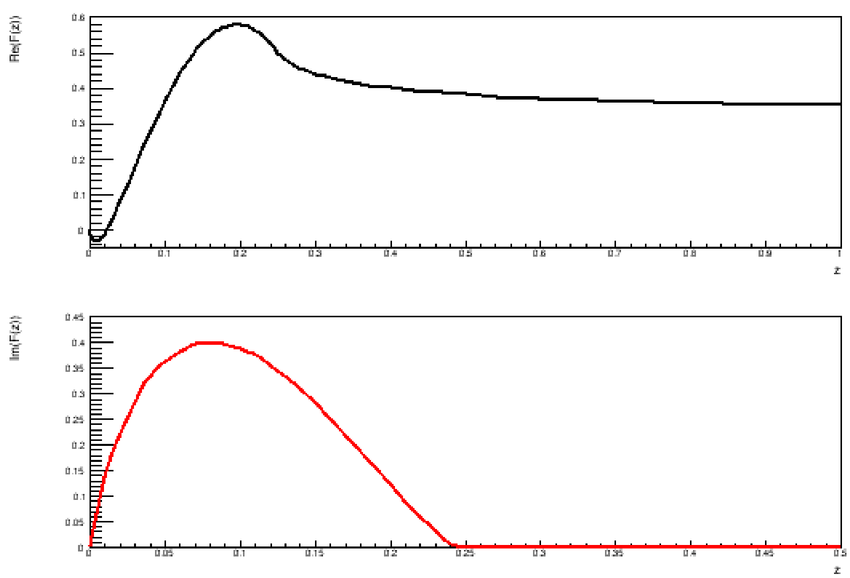


Figure 7: *Real and imaginary part of the function $\mathcal{F}(z)$ with respect to $z = m_q^2/M_H^2$*

real and imaginary parts of $\mathcal{F}(z)$ with respect to z (see sec. 9.3). It can be shown that this function has the limit $1/3$ when $z \to \infty$. To show that let us come back to eq. (9.14) which gives the integral representation of the function $J(z)$, when $z \to \infty$, the denominator cannot vanish so we can take safely the limit $\epsilon \to 0$ and write:

$$\frac{1}{1 - y^2\, x\,(1-x)/z} \simeq 1 + y^2\, x\,(1-x)/z \; .$$

So in this limit, the function $J(z)$ behaves as:

$$\begin{aligned} J(z) &\simeq \int_0^1 dx \int_0^1 y\, dy + \frac{1}{z} \int_0^1 dx\, x\,(1-x) \int_0^1 dy\, y^3 \\ &\simeq \frac{1}{2} + \frac{1}{24\, z}\ , \end{aligned} \tag{9.20}$$

and therefor the function $\mathcal{F}(z) \to 1/3$ when $z \to \infty$. Since the amplitude T_2 can be obtained from the amplitude T_1 by changing $\epsilon(p_1), p_1 \leftrightarrow \epsilon(p_2), p_2$, it is clear from eq. (9.18) that $T_2 = T_1$. So the total amplitude $T = T_1 + T_2$ is:

$$T = -\frac{i}{4\,\pi}\, \frac{\alpha_s\, e}{\sin\theta_W}\, \frac{M_H^2}{M_W}\, \delta^{ab}\, \epsilon_\mu^a(p_1)\, \epsilon_\nu^b(p_2) \left(g^{\mu\,\nu} - \frac{2\, p_1^\nu\, p_2^\mu}{M_H^2} \right) \mathcal{F}\left(\frac{m_q^2}{M_H^2} \right)\ , \tag{9.21}$$

where the notation $\alpha_s = g_s^2/(4\,\pi)$ has been introduced for the strong interaction coupling. Note that in eq. (9.21), if we replace $\epsilon(p_1)$ (respectively $\epsilon(p_2)$) by p_1 (resp. p_2), the amplitude T vanishes because:

$$p_{1\,\mu} \left(g^{\mu\,\nu} - \frac{2\, p_1^\nu\, p_2^\mu}{M_H^2} \right) = 0$$

Let us now compute the modulus squared of the amplitude averaging over the initial spins and colours:

$$|\bar{T}|^2 = \frac{1}{4\,(N^2-1)^2} \sum_{\text{polarisations}} \sum_{\text{colours}} |T|^2$$

For the average over the initial spins, we have to compute something like:

$$\begin{aligned} S &= \sum_{\text{polarisations}} \epsilon_\mu^a(p_1)\, \epsilon_\nu^b(p_2)\, \epsilon_\rho^{c\,\star}(p_1)\, \epsilon_\sigma^{d\,\star}(p_2) \left(g^{\mu\,\nu} - \frac{2\, p_1^\nu\, p_2^\mu}{M_H^2} \right) \left(g^{\rho\,\sigma} - \frac{2\, p_1^\sigma\, p_2^\rho}{M_H^2} \right) \\ &= \delta^{a\,c}\, \delta^{b\,d}\, (-g_{\mu\,\rho})\, (-g_{\nu\,\sigma}) \left(g^{\mu\,\nu} - \frac{2\, p_1^\nu\, p_2^\mu}{M_H^2} \right) \left(g^{\rho\,\sigma} - \frac{2\, p_1^\sigma\, p_2^\rho}{M_H^2} \right) \\ &= 2\, \delta^{a\,c}\, \delta^{b\,d} \end{aligned}$$

Note that we have taken $\sum_{\text{pol.}} \epsilon_\mu^a(p_1)\, \epsilon_\rho^{c\,\star}(p_1) = -\delta^{a\,c}\, g^{\mu\,\rho}$ this is justified in this case because the replacement of $\epsilon^\mu(p_1)$ by p_1^μ ($\epsilon^\nu(p_2)$ by p_2) gave zero. Now, for the average over the initial colours, we have to compute:

$$C = \sum_{a,b,c,d} \delta^{a\,b}\, \delta^{c\,d}\, \delta^{a\,c}\, \delta^{b\,d} = \sum_a \delta^{a\,a} = N^2 - 1$$

where N is the number of colours. Finally we obtain for the matrix element squared:

$$|\bar{T}|^2 = \frac{\alpha_s^2\, \alpha\, M_H^4}{8\,\pi\,(N^2-1)\,(\sin\theta_W\, M_W)^2} \left| \mathcal{F}\left(\frac{m_q^2}{M_H^2} \right) \right|^2\ , \tag{9.22}$$

where the fine structure constant $\alpha = e^2/4\pi$ is introduced. So far, we considered only one quark flavour in the loop, in principle we need to sum over all the possible flavours of quarks so that eq. (9.5) becomes:

$$\boxed{\begin{aligned}\sigma_H = & \frac{1}{8\,(N^2-1)}\,\frac{\alpha_s^2\,\alpha\,M_H^2}{(\sin\theta_W\,M_W)^2\,S}\left(\sum_{q=d,u,s,c,b,t}\left|\mathcal{F}\left(\frac{m_q^2}{M_H^2}\right)\right|^2\right)\\ & \times\int_{M_H^2/S}^{1}\frac{dx_1}{x_1}\,F_g^P(x_1,M^2)\,F_g^P\left(\frac{M_H^2}{x_1\,S},M^2\right)\end{aligned}} \tag{9.23}$$

In practice, we can content ourselves to keep only the top quark since the function $\mathcal{F}$ is vanishingly small for other quark species. The scale M which appears in the partonic densities of eq. (9.23) must be taken of the order of the Higgs boson mass (M_H) because this is the only "hard" energy scale (much greater than Λ_{QCD}) which remains.

9.2 Function Li_2

The Li_2 function is defined as:

$$\mathrm{Li}_2(z) = -\int_0^y dt\,\frac{\ln(1-t)}{t} = -\int_0^1 dt\,\frac{\ln(1-z\,t)}{t} \tag{9.24}$$

with z complex. From its definition, the function Li_2 has a cut in the complex plan on the real axis $[1,\infty[$. Furthermore, we have the following property:

$$\mathrm{Li}_2(1) = \frac{\pi^2}{6} = \sum_{k=1}^{\infty}\frac{1}{k^2}.$$

In the case where z has an infinitesimal imaginary part $z = x \pm i\,\epsilon$ and a real part $x > 1$, from the definition of the function Li_2, we can show that:

$$\mathrm{Li}_2\,(x \pm i\,\epsilon) \overset{\epsilon \to 0}{=} -\mathrm{Li}_2\left(\frac{1}{x}\right) - \frac{1}{2}\ln^2\left(\frac{1}{x}\right) + \frac{\pi^2}{3} \mp i\,\pi\,\ln\left(\frac{1}{x}\right). \tag{9.25}$$

This equation gives us the prescription for $x > 1$. More generally, if z is a complex number with a non vanishing imaginary part (it is always the case if we carefuly keep track of the small imaginary part ϵ), we have the following relations:

$$\mathrm{Li}_2\left(\frac{1}{z}\right) = -\mathrm{Li}_2(z) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2(-z) \tag{9.26}$$

$$\mathrm{Li}_2(1-z) = -\mathrm{Li}_2(z) + \frac{\pi^2}{6} - \ln(1-z)\,\ln(z)\,. \tag{9.27}$$

9.3 Different rewritting of the function $J(z)$

We can apply the relations in the section above to simplify $J(z)$ (eq. (9.16)). Let us start with the case where $z \le 1/4$. In this case the real parts of x_1 and x_2 are between 0 and 1. We can use eq. (9.25) and we obtain that:

$$\begin{aligned}
\mathrm{Li}_2\left(\frac{1}{x_1}\right) + \mathrm{Li}_2\left(\frac{1}{x_2}\right) &= -\mathrm{Li}_2(y) - \mathrm{Li}_2(1-y) - \frac{1}{2}\ln^2(y) - \frac{1}{2}\ln^2(1-y) \\
&\quad + \frac{2\,\pi^2}{3} - i\,\pi\left(\ln(1-y) - \ln(y)\right) , \qquad (9.28)
\end{aligned}$$

with y the real part of x_1, $y = 1/2\,(1+\sqrt{1-4\,z})$. Then, we can apply the relation (9.27) to the equation (9.28), this gives:

$$\begin{aligned}
\mathrm{Li}_2\left(\frac{1}{x_1}\right) + \mathrm{Li}_2\left(\frac{1}{x_2}\right) &= -\frac{1}{2}\ln^2(y) - \frac{1}{2}\ln^2(1-y) + \ln(y)\,\ln(1-y) \\
&\quad + \frac{\pi^2}{2} - i\,\pi\left(\ln(1-y) - \ln(y)\right) \\
&= -\frac{1}{2}\left(\ln(1-y) - \ln(y)\right)^2 + \frac{\pi^2}{2} - i\,\pi\left(\ln(1-y) - \ln(y)\right) . \qquad (9.29)
\end{aligned}$$

As $0 \le y \le 1$, we can group the logarithms and we get:

$$\begin{aligned}
\mathrm{Li}_2\left(\frac{1}{x_1}\right) + \mathrm{Li}_2\left(\frac{1}{x_2}\right) &= -\frac{1}{2}\left[\ln\left(\frac{1}{y} - 1\right) + i\,\pi\right]^2 \\
&= -\frac{1}{2}\ln^2\left(1 - \frac{1}{x_1}\right) . \qquad (9.30)
\end{aligned}$$

In the case where $z > 1/4$, x_1 and x_2 are complex conjugate but with an imaginary part which is not infinitesimal. We will use the relation (9.26) to write that:

$$\mathrm{Li}_2\left(\frac{1}{x_1}\right) + \mathrm{Li}_2\left(\frac{1}{x_2}\right) = -\mathrm{Li}_2(x_1) - \mathrm{Li}_2(1-x_1) - \frac{1}{2}\ln^2(-x_1) - \frac{1}{2}\ln^2(x_1-1) - \frac{\pi^2}{3} . \qquad (9.31)$$

Then, applying eq. (9.27), the sum of the dilogarithms becomes:

$$\begin{aligned}
\mathrm{Li}_2\left(\frac{1}{x_1}\right) + \mathrm{Li}_2\left(\frac{1}{x_2}\right) &= -\frac{1}{2}\ln^2(-x_1) - \frac{1}{2}\ln^2(x_1-1) - \frac{\pi^2}{2} \\
&\quad + \ln(x_1)\,\ln(1-x_1) \\
&= -\frac{1}{2}\left(\ln(x_1-1) - \ln(-x_1)\right)^2 - \frac{\pi^2}{2} \\
&\quad - \ln(x_1-1)\,\ln(-x_1) + \ln(x_1)\,\ln(1-x_1) . \qquad (9.32)
\end{aligned}$$

Let us remark that x_1 has a real part and an imaginary part which are both positive, it lies then in the first quadrand. With the convention that the cut of the logarithm is along the negative real axis, then the phase of a complex number in the main Rieman sheet is between $-\pi$ and π, if x_1 is parametrised like $\rho\, e^{i\,\theta}$ then $-x_1 = \rho\, e^{i\,(\theta-\pi)}$ and so the relation between the logarithms of x_1 and $-x_1$ is:

$$\ln(-x_1) = \ln(x_1) - i\,\pi\ .$$

In the same way, $1 - x_1$ has a positive real part and a negative imaginary part, so if $1 - x_1 = \rho\, e^{i\,\theta}$ then $x_1 - 1 = \rho\, e^{i\,(\theta+\pi)}$ and we have that:

$$\ln(x_1 - 1) = \ln(1 - x_1) + i\,\pi\ .$$

Using that, we write:

$$\ln(x_1)\, \ln(1 - x_1) = \ln(-x_1)\, \ln(x_1 - 1) + i\,\pi\,(\ln(x_1 - 1) - \ln(-x_1)) + \pi^2\ , \tag{9.33}$$

so the sum of the two dilogarithms can be written:

$$\begin{aligned} \mathrm{Li}_2\left(\frac{1}{x_1}\right) + \mathrm{Li}_2\left(\frac{1}{x_2}\right) &= -\frac{1}{2}\,(\ln(x_1 - 1) - \ln(-x_1))^2 + \frac{\pi^2}{2} + i\,\pi\,(\ln(x_1 - 1) - \ln(-x_1)) \\ &= -\frac{1}{2}\,\left[\ln(x_1 - 1) - \ln(-x_1) - i\,\pi\right]^2\ . \end{aligned} \tag{9.34}$$

The term $i\,\pi$ can be reabsorbed by writing $\ln(1-x_1)$ instead of $\ln(x_1-1)$ and remarking that $1-x_1 = x_2$ and $-x_1$ have a same sign imaginary part, then we finally get:

$$\mathrm{Li}_2\left(\frac{1}{x_1}\right) + \mathrm{Li}_2\left(\frac{1}{x_2}\right) = -\frac{1}{2}\,\ln^2\left(1 - \frac{1}{x_1}\right)\ . \tag{9.35}$$

Thus, $J(z)$ can be simplified such that only the logarithmic function is used for buth cases:

$$J(z) = -\frac{z}{2}\left\{\begin{array}{ll} \ln^2\left(\frac{\sqrt{1-4\,z}-1+i\,\epsilon}{\sqrt{1-4\,z}+1+i\,\epsilon}\right) & z \le 1/4 \\ \ln^2\left(\frac{i\,\sqrt{4\,z-1}-1}{i\,\sqrt{4\,z-1}+1}\right) & z > 1/4 \end{array}\right. . \tag{9.36}$$

To conclude these technical remarks, we show how to rewrite $J(z)$ to make easy the comparison with the results which can be found in the litterature. In the case $z \le 1/4$, it is easy to show that:

$$\begin{aligned} \ln\left(\frac{\sqrt{1-4\,z}-1+i\,\epsilon}{\sqrt{1-4\,z}+1+i\,\epsilon}\right) &= \ln\left(\frac{\sqrt{1-4\,z}-1}{\sqrt{1-4\,z}+1} + i\,\epsilon\right) \\ &= -\ln\left(\frac{1+\sqrt{1-4\,z}}{1-\sqrt{1-4\,z}}\right) + i\,\pi\ . \end{aligned} \tag{9.37}$$

For the case $z > 1/4$, we write:

$$\ln\left(\frac{i\,\sqrt{4\,z-1}-1}{i\,\sqrt{4\,z-1}+1}\right) = \ln\left(\frac{\sqrt{1-\frac{1}{4\,z}} + i\,\sqrt{\frac{1}{4\,z}}}{\sqrt{1-\frac{1}{4\,z}} - i\,\sqrt{\frac{1}{4\,z}}}\right)\ . \tag{9.38}$$

Remarking that the complex number $\sqrt{1-1/4\,z}+i\,\sqrt{1/4\,z}$ has a modulus which is equal to 1 and it lies in the first quadrand, we show:

$$\begin{aligned}
\ln\left(\frac{i\,\sqrt{4\,z-1}-1}{i\,\sqrt{4\,z-1}+1}\right) &= \ln\left(\left(\sqrt{1-\frac{1}{4\,z}}+i\,\sqrt{\frac{1}{4\,z}}\right)^2\right) \\
&= 2\,\ln\left(\sqrt{1-\frac{1}{4\,z}}+i\,\sqrt{\frac{1}{4\,z}}\right) \\
&= 2\,i\,\arcsin\left(\sqrt{\frac{1}{4\,z}}\right)\quad .
\end{aligned} \tag{9.39}$$

Then, the function $J(z)$ becomes:

$$J(z)=\frac{4\,z}{2}\left\{\begin{array}{cc} -\frac{1}{4}\left(\ln\left(\frac{1+\sqrt{1-4\,z}}{1-\sqrt{1-4\,z}}\right)-i\,\pi\right)^2 & z\le 1/4 \\ \arcsin^2\left(\sqrt{\frac{1}{4\,z}}\right) & z>1/4 \end{array}\right. \quad . \tag{9.40}$$