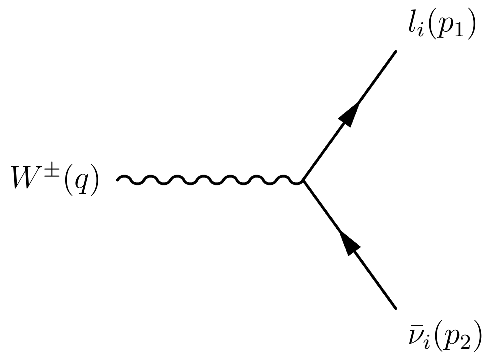
1 Decay of the W boson into leptons

The diagram representing this decay is:



The amplitude related to this diagram is given by:

$$M = \bar{u}(p_1) \left(-i \frac{g}{2\sqrt{2}} \gamma^\mu (1-\gamma^5) \right) v(p_2)\, \epsilon_\mu(q) \tag{1}$$

with $q = p_1 + p_2$. We sum on the final polarizations and we average on the initial polarizations (3 states), the square of the amplitude becomes:

$$|\overline{M}|^2 = \frac{1}{3}\frac{g^2}{8} Tr\left[(\not{p}_1 + m_e)\, \gamma^\mu (1-\gamma^5)\, (\not{p}_2 - m_\nu)\, \gamma^\nu (1-\gamma^5) \right] \sum_{pol} \epsilon_\mu(q)\, \epsilon^*_\nu(q) \tag{2}$$

Remembering that:

$$\begin{aligned} (1-\gamma^5)(1-\gamma^5) &= 2(1-\gamma^5) \\ (1-\gamma^5)(1+\gamma^5) &= 0 \end{aligned}$$

and that the traces of an odd number of matrices γ and of γ^5 times an odd number of matrices γ are zero, the equation (2) reads:

$$|\overline{M}|^2 = \frac{1}{3}\frac{g^2}{4} Tr\left[\not{p}_1\, \gamma^\mu (1-\gamma^5)\, \not{p}_2\, \gamma^\nu \right] \left(-g^{\mu\nu} + \frac{q^\mu\, q^\nu}{M_w^2} \right) \tag{3}$$

The part proportional to γ^5 in the trace gives zero because it is contracted with a symmetric tensor in μ and ν, so we have to calculate:

$$|\overline{M}|^2 = \frac{1}{3}\frac{g^2}{4} \left\{ 2\, Tr\left[\not{p}_1\, \not{p}_2 \right] + \frac{1}{M_w^2} Tr\left[\not{p}_1\, (\not{p}_1 + \not{p}_2)\, \not{p}_2\, (\not{p}_1 + \not{p}_2) \right] \right\} \tag{4}$$

Using the formulas of the traces of the matrices γ, we find:

$$|\overline{M}|^2 = \frac{g^2}{3} \left[M_w^2 - \frac{1}{2}(m_e^2 + m_\nu^2) - \frac{1}{2} \frac{(m_e^2 - m_\nu^2)^2}{M_w^2} \right] \tag{5}$$

The decay rate of a particle of mass M into n body, in the frame where this particle is at rest, is given by:

$$d\Gamma = \frac{1}{2\,M} |\overline{M}|^2\, d\Phi_n(P; p_1, \cdots, p_n) \tag{6}$$

with

$$d\Phi_n(P;p_1,\cdots,p_n) = (2\,\pi)^4\,\delta^4(P-\sum_{i=1}^{n} p_i)\prod_{i=1}^{n}\frac{d^3p_i}{(2\,\pi)^3\,2\,E_i}$$

In our case, we have two bodies in the final state ($n=2$):

$$d\Gamma = \frac{1}{2\,M_w}\,|\overline{M}|^2\,(2\,\pi)^4\,\delta^4(q-p_1-p_2)\,\frac{d^3p_1}{(2\,\pi)^3\,2\,E_1}\,\frac{d^3p_2}{(2\,\pi)^3\,2\,E_2} \tag{7}$$

We work in the frame where the W boson is at rest:

$$q = (M_w,0,0,0) \to \begin{cases} E_1+E_2 &= M_w \\ \vec{p_1}+\vec{p_2} &= \vec{0} \end{cases}$$

as there is no initial direction, we can orient the axes of the frame so that $\vec{p_1}$ is along OZ

$$\vec{p_1} = (0,0,p_z) \quad \vec{p_2} = (0,0,-p_z)$$

Replacing $\int d^3p_2/2/E_2$ by $\int d^4p_2\,\delta^+(p_2^2-m_\nu^2)$ and integrating with respect to p_2, we find:

$$\begin{aligned} d\Gamma &= \frac{1}{2\,M_w}\,\frac{1}{(2\,\pi)^2}\int\frac{d^3p_1}{2\,E_1}\,\delta^+\left((q-p_1)^2-m_\nu^2\right)\,|\overline{M}|^2 \\ &= \frac{1}{16\,\pi^2\,M_w}\int\frac{p_z^2}{\sqrt{p_z^2+m_e^2}}\,dp_z\,d\Omega_1\,\delta^+\left(M_w^2+m_e^2-m_\nu^2-2\,M_w\sqrt{p_z^2+m_e^2}\right)\,|\overline{M}|^2 \end{aligned} \tag{8}$$

where Ω_1 is the solid angle around the direction of $\vec{p_1}$.

The integration over p_z is of the type:

$$I = \int dp_z\,f(p_z)\,\delta^+\left(M_w^2+m_e^2-m_\nu^2-2\,M_w\sqrt{p_z^2+m_e^2}\right)$$

The argument of the Dirac distribution δ vanishes for:

$$p_z = p_z^0 = \pm\frac{\sqrt{\lambda(M_w^2,m_e^2,m_\nu^2)}}{2\,M_w}$$

where:

$$\lambda(x,y,z) = x^2+y^2+z^2-2\,x\,y-2\,x\,z-2\,y\,z$$

so I reads:

$$I = \frac{f(p_z^0)\,\sqrt{p_z^0+m_e^2}}{2\,M_w\,p_z^0}$$

and thus:

$$d\Gamma = \frac{1}{16\,\pi^2\,M_w}\,\frac{\sqrt{\lambda(M_w^2,m_e^2,m_\nu^2)}}{4\,M_w^2}\int d\Omega_1|\overline{M}|^2 \tag{9}$$

The integral on the solid angle of the fermion is trivial because $|\overline{M}|^2$ depends only on the masses, it gives a factor $4\,\pi$. So we end with the following result:

$$\Gamma = \frac{g^2}{48\,\pi M_w}\,\frac{\sqrt{\lambda(M_w^2,m_e^2,m_\nu^2)}}{M_w^2}\left\{M_w^2-\frac{1}{2}\,(m_e^2+m_\nu^2)-\frac{1}{2}\,\frac{(m_e-m_\nu^2)^2}{M_w^2}\right\} \tag{10}$$

or in term of G_F (remember that $g^2=\frac{8\,G_F\,M_w^2}{\sqrt{2}}$):

$$\Gamma = \frac{\sqrt{2}\,G_F\,M_w}{12\,\pi}\,\frac{\sqrt{\lambda(M_w^2,m_e^2,m_\nu^2)}}{M_w^2}\left\{M_w^2-\frac{1}{2}\,(m_e^2+m_\nu^2)-\frac{1}{2}\,\frac{(m_e-m_\nu^2)^2}{M_w^2}\right\} \tag{11}$$

If we neglect the masses of leptons with respect to the mass of W, we then get:

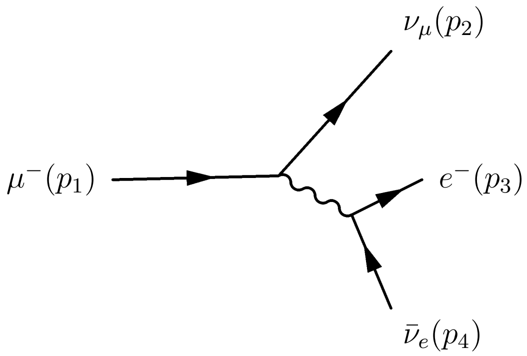
$$\Gamma = \frac{\sqrt{2}\,G_F\,M_w^3}{12\,\pi} \tag{12}$$

2 Muon decay

The muon decays weakly according to the reaction:

$$\mu^-(p_1) \to \nu_\mu(p_2) + e^-(p_3) + \bar{\nu}_e(p_4)$$

The diagram representing this decay is:



whose amplitude is:

$$M = \bar{u}(p_2)\left(-i\,\frac{g}{2\sqrt{2}}\,\gamma^\mu\,(1-\gamma^5)\right)\,u(p_1)\,\bar{u}(p_3)\,\left(-i\,\frac{g}{2\sqrt{2}}\,\gamma^\nu\,(1-\gamma^5)\right)\,v(p_4)\,\frac{-i(g_{\mu\nu}-q_\mu\,q_\nu/M_w^2)}{q^2-M_w^2} \tag{13}$$

or else:

$$\begin{aligned} M &= i\,\frac{g^2}{8}\,\frac{1}{q^2-M_w^2}\,\Big[\bar{u}(p_2)\,\gamma^\mu\,(1-\gamma^5)\,u(p_1)\,\bar{u}(p_3)\,\gamma^\nu\,(1-\gamma^5)\,v(p_4) \\ &\quad -\frac{1}{M_w^2}\bar{u}(p_2)\,\not{q}\,(1-\gamma^5)\,u(p_1)\,\bar{u}(p_3)\,\not{q}\,(1-\gamma^5)\,v(p_4)\Big] \end{aligned} \tag{14}$$

where q is the 4-momentum of the W boson exchanged: $q = p_3 + p_4 = p_1 - p_2$.

We take $m_{\nu_e} = 0$ and $m_{\nu_\mu} = 0$. This is justified by the fact that first and second generation neutrinos have very small masses of the order of electron volts. Note that in the equation (14), the second term is not zero because the current $\bar{u}(p_2)\,\gamma^\mu\,(1-\gamma^5)\,u(p_1)$ is not conserved with massive fermions:

$$\begin{aligned} \bar{u}(p_2)\,(\not{p}_1 - \not{p}_2)\,(1-\gamma^5)\,u(p_1) &= m_\mu\,\bar{u}(p_2)\,(1+\gamma^5)\,u(p_1) \\ \bar{u}(p_3)\,(\not{p}_3 + \not{p}_4)\,(1-\gamma^5)\,v(p_4) &= m_e\,\bar{u}(p_3)\,(1-\gamma^5)\,v(p_4) \end{aligned}$$

Thus, the amplitude reads:

$$\begin{aligned} M &= i\,\frac{g^2}{8}\,\frac{1}{q^2-M_w^2}\,\Big[\bar{u}(p_2)\,\gamma^\mu\,(1-\gamma^5)\,u(p_1)\,\bar{u}(p_3)\,\gamma^\nu\,(1-\gamma^5)\,v(p_4) \\ &\quad -\frac{m_\mu\,m_e}{M_w^2}\,\bar{u}(p_2)\,(1+\gamma^5)\,u(p_1)\,\bar{u}(p_3)\,(1-\gamma^5)\,v(p_4)\Big] \end{aligned} \tag{15}$$

The amplitude squared, averaged over the spin initial state and summed on the final spin states, can be written as:

$$
\begin{aligned}
|\overline{M}|^2 &= \frac{g^4}{64}\,\frac{1}{(q^2-M_w^2)^2}\,\frac{1}{2}\Big[Tr[\not p_2\,\gamma^\mu\,(1-\gamma^5)\,(\not p_1+m_\mu)\,(1+\gamma^5)\,\gamma^\nu] \\
&\times Tr[(\not p_3+m_e)\,\gamma_\mu\,(1-\gamma^5)\,\not p_4\,(1+\gamma^5)\,\gamma_\nu] \\
&+\frac{m_\mu^2\,m_e^2}{M_w^4}\,Tr[\not p_2\,(1+\gamma^5)\,(\not p_1+m_\mu)\,(1-\gamma^5)]\,Tr[(\not p_3+m_e)\,(1-\gamma^5)\,\not p_4\,(1+\gamma^5)] \\
&-2\,\frac{m_\mu\,m_e}{M_w^2}\,Tr[\not p_2\,\gamma^\mu\,(1-\gamma^5)\,(\not p_1+m_\mu)\,(1-\gamma^5)]\,Tr[(\not p_3+m_e)\,\gamma_\mu\,(1-\gamma^5)\,\not p_4\,(1+\gamma^5)]\Big]
\end{aligned} \tag{16}
$$

Remembering that:

$$
\begin{aligned}
(1-\gamma^5)\,(1-\gamma^5) &= 2\,(1-\gamma^5) \\
(1-\gamma^5)\,(1+\gamma^5) &= 0
\end{aligned}
$$

and that the traces of an odd number of matrices γ and of γ^5 times an odd number of matrices γ are zero, the equation (16) becomes:

$$
\begin{aligned}
|\overline{M}|^2 &= \frac{g^4}{8}\,\frac{1}{(q^2-M_w^2)^2}\Big[4\,(p_2^\mu\,p_1^\nu+p_2^\nu\,p_1^\mu-g^{\mu\nu}\,p_1.p_2+i\,\epsilon^{\rho\mu\sigma\nu}\,p_{2\,\rho}\,p_{1\,\sigma}) \\
&\times(p_{3\,\mu}\,p_{4\,\nu}+p_{3\,\nu}\,p_{4\,\mu}-g_{\mu\nu}\,p_3.p_4+i\,\epsilon_{\alpha\,\mu\,\beta\,\nu}\,p_3^\alpha\,p_4^\beta) \\
&+4\,\frac{m_\mu^2\,m_e^2}{M_w^4}\,p_1.p_2\,p_3.p_4-8\,\frac{m_\mu^2\,m_e^2}{M_w^2}\,p_2.p_4\Big]
\end{aligned} \tag{17}
$$

Contracting over repeated indices yields:

$$
|\overline{M}|^2 = \frac{g^4}{2}\,\frac{1}{(q^2-M_w^2)^2}\left[4\,p_2.p_3\,p_1.p_4+\frac{m_\mu^2\,m_e^2}{M_w^4}\,(p_1.p_2\,p_3.p_4-2\,M_w^2\,p_2.p_4)\right] \tag{18}
$$

We will now calculate the transition rate and perform the integration on the phase space. The latter is given by:

$$
\begin{aligned}
d\Gamma &= \frac{1}{2\,m_\mu}\,\frac{1}{(2\,\pi)^5}\,\delta^4(p_1-p_2-p_3-p_4)\,\prod_{i=2}^{4}\frac{d^3p_i}{(2\,\pi)^3\,2\,E_i}\,|\overline{M}|^2 \\
&= \frac{1}{2\,m_\mu}\,\frac{1}{(2\,\pi)^5}\,\frac{d^3p_2}{2\,E_2}\,\frac{d^3p_3}{2\,E_3}\,\delta^+(m_e^2+m_\mu^2-2\,p_1.p_2-2\,p_1.p_3+2\,p_2.p_3)\,|\overline{M}|^2
\end{aligned} \tag{19}
$$

We will work in the frame where the muon is at rest, that is to say: $p_1 = m_\mu\,(1,0,0,0)$. We can orient the axes of this frame in such a way that: $p_2 = E_2\,(1,0,0,1)$ et $p_3 = (E_3, |p_3|\,\sin(\theta), 0, |p_3|\,\cos(\theta))$ with $E_3^2 - |p_3|^2 = m_e^2$. Equation (19) becomes:

$$
\begin{aligned}
d\Gamma &= \frac{1}{2\,m_\mu}\,\frac{1}{(2\,\pi)^5}\,\frac{1}{2}\,E_2\,dE_2\,d\Omega_2\,\frac{|p_3|^2}{2\,E_3}\,d|p_3|\,d\Omega_3 \\
&\quad\delta^+(m_e^2+m_\mu^2-2\,m_\mu\,E_3-2\,E_2\,(m_\mu-E_3+|p_3|\,\cos(\theta)))\,|\overline{M}|^2
\end{aligned} \tag{20}
$$

To simplify the calculations, we will make additional approximations, namely that M_w is large with respect to all energy scales and that m_e is negligible with respect to these same scales. This is justified by the fact that $M_w \gg m_\mu \gg m_e$.

We will use the constraint of the Dirac distribution to integrate on $\cos(\theta)$, so:

$$d\Gamma \;=\; \frac{\pi^2}{m_\mu}\,\frac{1}{(2\,\pi)^5}\,E_3\,dE_3\,d\cos(\theta)\,E_2\,dE_2\,\frac{1}{2\,E_2\,E_3}\,\delta^+(\cos(\theta)-C)\,|\overline{M}|^2 \tag{21}$$

with

$$C = \frac{m_\mu^2 - 2\,m_\mu\,(E_2+E_3) + 2\,E_2\,E_3}{2\,E_2\,E_3}$$

This leads to:

$$d\Gamma \;=\; \frac{\pi^2}{2\,m_\mu}\,\frac{1}{(2\,\pi)^5}\,dE_3\,dE_2\;|\overline{M}|^2\big|_{\cos(\theta)=C} \tag{22}$$

In the approximation where the mass of the electron is neglected, and where $q^2 \ll M_w^2$, the square of the amplitude taken at $\cos(\theta) = C$ becomes:

$$|\overline{M}|^2_{\cos(\theta)=C} \;=\; \frac{g^4\,m_\mu^2}{2\,M_w^4}\left(6\,m_\mu\,E_2 + 6\,m_\mu\,E_3 - 2\,m_\mu^2 - 4\,(E_2+E_3)^2\right) \tag{23}$$

But we have the constraint that $-1 \le C \le 1$ because $\cos(\theta)$ must range between -1 and 1. The limit $C \le 1$ implies that $E_2+E_3 \ge m_\mu/2$ and the lower bound that $(m_\mu - 2\,E_2)\,(m_\mu - 2\,E_3) \ge 0$.

To lighten the discussion, it is better to introduce the reduced energy $z_i = 2\,E_i/m_\mu$ On the other hand, in our approximation, from energy-momentum conservation $p_1 = p_2 + p_3 + p_4$ we get:

$$\begin{aligned} 2\,p_2.p_3 &= p_1^2 - 2\,p_1.p_4 = m_\mu^2 - 2\,m_\mu\,E_4 \\ 2\,p_2.p_4 &= p_1^2 - 2\,p_1.p_3 = m_\mu^2 - 2\,m_\mu\,E_3 \\ 2\,p_3.p_4 &= p_1^2 - 2\,p_1.p_2 = m_\mu^2 - 2\,m_\mu\,E_2 \end{aligned} \tag{24}$$

But the dot product of two time-like quadri-vectors is positive. This implies that $E_i \le m_\mu/2$ for $i = 2,3,4$ and thus $0 \le z_i \le 1$ $i = 2,3,4$. In terms of these new variables, the contraints read: $z_2 + z_3 \ge 1$ and $(1-z_2)\,(1-z_3) \ge 0$. The latter is always verified and the decay rate becomes:

$$\begin{aligned} \Gamma \;=\; & \frac{m_\mu^5}{32}\,\frac{1}{(2\,\pi)^3}\,\frac{g^4}{2\,M_W^4}\int_0^1 dz_3 \int_{1-z_3}^1 dz_2 \\ & \times \left(3\,z_2 + 3\,z_3 - 2 - z_2^2 - z_3^2 - 2\,z_2\,z_3\right) \end{aligned} \tag{25}$$

The integration over z_2 is performed easily and the decay rate reads:

$$\Gamma \;=\; \frac{m_\mu^5}{32}\,\frac{1}{(2\,\pi)^3}\,\frac{g^4}{12\,M_W^4}\int_0^1 dz_3\,\left[3\,z_3^2 - 2\,z_3^3\right] \tag{26}$$

The integral on z_3 gives 1/2 and by expressing the coupling constant in term of G_F ($g^4 = 32\,M_w^4\,G_F^2$), the integrated transition rate therefore becomes:

$$\Gamma \;=\; \frac{G_F^2\,m_\mu^5}{192\,\pi^3} \tag{27}$$

Returning to the MKSI unit system, the muon life time is given by:

$$\tau = \frac{\hbar}{\Gamma}$$

From the “Particle Data Book”, if we take $\hbar = 6.58211899\,10^{-25}$ MeV s, $G_F = 1.16637\,10^{-5}$ GeV, $m_\mu = 0.10565837$ GeV, we find:

$$\tau = 2.1873\,10^{-6}\ \mathrm{s}$$

The experimental measurements of the muon life time give (see PDB) $\tau = 2.197034 \pm 0.000004\,10^{-6}$ s, keeping in mind our approximations, we have a satisfactory agreement with experience.